QUANTUM SUPERALGEBRA REPRESENTATIONS ON COHOMOLOGY GROUPS OF NON-COMMUTATIVE BUNDLES

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Abstract. Quantum homogeneous supervector bundles arising from the quantum general linear supergroup are studied. The space of holomorphic sections is promoted to a left exact covariant functor from a category of modules over a quantum parabolic sub-supergroup to the category of locally finite modules of the quantum general linear supergroup. The right derived functors of this functor provides a form of Dolbeault cohomology for quantum homogeneous supervector bundles. We explicitly compute the cohomology groups, which are given in terms of well understood modules over the quantized universal enveloping algebra of the general linear superalgebra.

1. Introduction

We follow the general philosophy of non-commutative geometry [3, 11] to study quantum homogeneous supervector bundles arising from the quantum general linear supergroup. Our starting point is the quantized universal enveloping algebra $\mathrm{U}_q(\mathfrak{gl}_{m|n})$ (see e.g., [20, 23]) of the complex general linear superalgebra $\mathfrak{gl}_{m|n}$ [7, 17]. As is well known, $\mathrm{U}_q(\mathfrak{gl}_{m|n})$ has the structure of a Hopf superalgebra [14, 13]. Thus its dual superspace $\mathrm{U}_q(\mathfrak{gl}_{m|n})^*$ acquires a natural associative superalgebraic structure. The subspace $\mathcal{A}(\mathfrak{gl}_{m|n})$ of $\mathrm{U}_q(\mathfrak{gl}_{m|n})^*$ spanned by all the matrix elements of the finite dimensional $\mathrm{U}_q(\mathfrak{gl}_{m|n})$-representations with integral weights forms a Hopf superalgebra, which may be considered as the superalgebra of functions on the quantum general linear supergroup. This Hopf superalgebra is closely related to the multi-parameter quantization of the general linear supergroup of [12], and obviously contains the Hopf superalgebra G_q of [21] as a Hopf sub-superalgebra.

For any given reductive quantum sub-superalgebra $\mathrm{U}_q(\mathfrak{l})$ of $\mathrm{U}_q(\mathfrak{gl}_{m|n})$, we consider the subspace $\mathcal{A}(\mathfrak{gl}_{m|n}, \mathfrak{l})$ of $\mathcal{A}(\mathfrak{gl}_{m|n})$ invariant with respect to left translations under $\mathrm{U}_q(\mathfrak{l})$. This subspace forms a sub-superalgebra of $\mathcal{A}(\mathfrak{gl}_{m|n})$. In the spirit of non-commutative geometry [3, 11], we shall regard $\mathcal{A}(\mathfrak{gl}_{m|n}, \mathfrak{l})$ as (the superalgebra of functions on) a quantum homogeneous superspace, and finite type projective $\mathcal{A}(\mathfrak{gl}_{m|n}, \mathfrak{l})$-modules as (spaces of global sections of) quantum supervector bundles on the quantum homogeneous superspace. As we shall see, such $\mathcal{A}(\mathfrak{gl}_{m|n}, \mathfrak{l})$-modules also admit a natural $\mathrm{U}_q(\mathfrak{gl}_{m|n})$-action. This provides an interesting link between the non-commutative geometry of the quantum supervector bundles and the representation theory of $\mathrm{U}_q(\mathfrak{gl}_{m|n})$. In the context of classical Lie groups, such a link is very well known and constitutes the subject of the Bott-Borel-Weil theory [2] (see [8] for a review on the algebraic theory). For Lie supergroups in the classical setting, the subject was studied by Penkov [15] (see also [16]).

We shall study a version of Dolbeault cohomology for quantum homogeneous supervector bundles. Our method is based on Zuckerman's algebraic theory of induced

Keywords. Quantum supergroups, derived functors, non-commutative vector bundles.
2000 Mathematics Subject Classification. Primary 17B37, 20G42, 17B10.

representations [4, 8], which was generalized to quantum groups by Andersen and co-workers in [1], and also to Lie superalgebras in [16]. We shall make essential use of results of [1].

We promote the space $\mathcal{S}$ of global sections and the space Γ of holomorphic sections of a quantum homogeneous supervector bundle to covariant functors from appropriate categories of modules over quantum sub-superalgebras of $\mathrm{U}_q(\mathfrak{gl}_{m|n})$ to the category of locally finite $\mathrm{U}_q(\mathfrak{gl}_{m|n})$-modules. The resultant functors are closely related to a quantum analogue of the Zuckerman functor, which will be introduced in Section 4. The derived functors of the 'holomorphic section functor' arising from Γ are the Dolbeault cohomology groups for quantum homogeneous supervector bundles which we seek for. When the quantum homogeneous supervector bundle $\mathcal{S}$ is induced by a finite dimensional irreducible module over a purely even reductive quantum subalgebra, or a finite dimensional dual Kac module (defined by (2.3)) over a general reductive quantum sub-superalgebra, we explicitly compute the cohomology groups in Theorem 5.2 and Theorem 5.3. The results resemble the classical Bott-Borel-Weil theorem [2] in that the cohomology is concentrated at one degree. However, the non-trivial cohomology groups are isomorphic to dual Kac modules over $\mathrm{U}_q(\mathfrak{gl}_{m|n})$, thus are not irreducible. The weight spaces of the cohomology groups are worked out.

The arrangement of the paper is as follows. Section 2 contains some background material on the general linear superalgebra and its quantized universal enveloping algebra. Section 3 introduces the notions of quantum homogeneous superspaces and quantum homogeneous supervector bundles on them, and study some basic properties of theirs. Section 4 studies induction functors. The Zuckerman functor is defined, and a version of Frobenius reciprocity is proven. Properties of various functors and module categories are also established, which are to be used in the next section. In Section 5 we first present the formulation Dolbeault cohomology for quantum homogeneous supervector bundles using results of Section 4, then compute the Dolbeault cohomology groups for bundles of interest.

2. Preliminaries

This section presents some preliminary material on the general linear superalgebra and its quantized universal enveloping algebra, which sets up the stage for the remainder of the paper. The section also serves to fix notations and conventions.

2.1. **The general linear superalgebra $\mathfrak{gl}_{m|n}$.** Through out the paper we shall denote by $\mathfrak{g}$ the complex general linear superalgebra $\mathfrak{gl}_{m|n}$ [7, 17]. Let $\mathbf{I}=\{1,2,\cdots,m+n\}$, and $\mathbf{I}'=\{1,2,\cdots,m+n-1\}$. We shall identify $\mathfrak{g}$ with the Lie superalgebra of $(m+n)\times(m+n)$-matrices, where the $\mathbb{Z}_2$-grading is specified as follows. Denote by e_{ab}, $a,b\in\mathbf{I}$, the $(m+n)\times(m+n)$-matrix unit with all entries being zero except that at the (a,b) position which is 1. We declare e_{ab} to be odd if $a\le m<b$ or $a>m\ge b$, and even otherwise. Then $\{e_{ab}|a,b\in\mathbf{I}\}$ forms a homogeneous basis of $\mathfrak{g}$. The maximal even subalgebra of $\mathfrak{g}$ will be denoted by $\mathfrak{g}_0$, which is equal to $\mathfrak{gl}_m\oplus\mathfrak{gl}_n$. Let $\mathfrak{g}_{+1}=\sum_{i\le m<r}\mathbb{C}e_{ir}$, and $\mathfrak{g}_{-1}=\sum_{i\le m<r}\mathbb{C}e_{ri}$. Then the odd subspace of $\mathfrak{g}$ is $\mathfrak{g}_{+1}\oplus\mathfrak{g}_{-1}$.

We fix the Borel subalgebra $\mathfrak{b}$ consisting of the upper triangular matrices, and take $\mathfrak{h}=\bigoplus_a\mathbb{C}E_{aa}$ as the Cartan subalgebra. Let $\{\epsilon_a\,|\,a\in\mathbf{I}\}$ be the basis of $\mathfrak{h}^*$ such that $\epsilon_a(E_{bb})=\delta_{ab}$. The space $\mathfrak{h}^*$ is equipped with a bilinear form $(\ ,\ ):\mathfrak{h}^*\times\mathfrak{h}^*\to\mathbb{C}$ such that $(\epsilon_a,\epsilon_b)=\left\{\begin{array}{ll}\delta_{ab}, & a\le m,\\ -\delta_{ab}, & a>m.\end{array}\right.$ We shall denote by $\mathfrak{h}^*_{\mathbb{Z}}$ the $\mathbb{Z}$-span of the ϵ_a. The set

of roots of $\mathfrak{g}$ is $\{\epsilon_a - \epsilon_b | a \neq b\}$, with $\epsilon_a - \epsilon_b$ being called odd if $a \leq m < b$ or $b \leq m < a$, and even otherwise. The set of the positive roots relative to the Borel subalgebra $\mathfrak{b}$ is $\{\epsilon_a - \epsilon_b \,|\, a < b\}$, and the set of simple roots is $\{\epsilon_a - \epsilon_{a+1} \,|\, a \in \mathbf{I}'\}$. An element $\lambda \in \mathfrak{h}^*$ is called dominant if $2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z}_+$, for all positive even roots of $\mathfrak{g}$. Denote by 2ρ the signed-sum of the positive roots of $\mathfrak{g}$. A $\lambda \in \mathfrak{h}^*$ is called $\mathfrak{g}$-regular if $(\lambda + \rho, \alpha) \neq 0$ for all even roots of $\mathfrak{g}$.

The elements of the following set $\{e_{a,a+1}, e_{a+1,a} | a \in \mathbf{I}'\} \cup \{e_{bb} | b \in \mathbf{I}\}$ generate $\mathfrak{g}$. We shall call a Lie sub-superalgebra $\mathfrak{r}$ of $\mathfrak{g}$ regular if there exist subsets $\Theta_\pm$ of $\mathbf{I}'$ and a subset Θ_0 of $\mathbf{I}$ such that $\mathfrak{r}$ is generated by the elements of the set $\{e_{aa} | a \in \Theta_0\} \cup \{e_{b,b+1} | b \in \Theta_+\} \cup \{e_{c+1,c} | c \in \Theta_-\}$. The Lie sub-superalgebra $\mathfrak{r}$ is called reductive if $\Theta_0 = \mathbf{I}$ and $\Theta_+ = \Theta_-$, and is called parabolic if $\Theta_0 = \mathbf{I}$, and either Θ_+ or Θ_- is equal to $\mathbf{I}'$. If $\mathfrak{r}$ is a parabolic Lie sub-superalgebra, then it contains the reductive Lie sub-superalgebra, called the Levi factor of $\mathfrak{r}$, generated by the elements of the set $\{e_{aa} | a \in \Theta_0\} \cup \{e_{b,b+1}, e_{b+1,b} | b \in \Theta_+ \cap \Theta_-\}$. Note that a parabolic Lie sub-superalgebra of $\mathfrak{g}$ necessarily contains $\mathfrak{b}$ or the opposite Borel subalgebra $\bar{\mathfrak{b}}$ spanned by the lower triangular matrices.

2.2. **The quantized universal enveloping algebra of $\mathfrak{gl}_{m|n}$.** Let q be an indeterminate, and denote by $\mathbb{C}(q)$ the field of complex rational functions in q. Set $q_a = \begin{cases} q, & a \leq m, \\ q^{-1}, & a > m. \end{cases}$ We define the quantized universal enveloping algebra $\mathrm{U}_q(\mathfrak{g})$ [20, 23, 21] of the general linear superalgebra $\mathfrak{g}$ to be the unital associative superalgebra over $\mathbb{C}(q)$ with the set of generators

$$\{E_{a,a+1},\ E_{a+1,a} \,|\, a \in \mathbf{I}'\} \cup \{K_b,\ K_b^{-1} \,|\, b \in \mathbf{I}\}$$

subject to the following relations

$$\begin{aligned}
K_a K_a^{-1} = 1, \qquad & K_a^{\pm 1} K_b^{\pm 1} = K_b^{\pm 1} K_a^{\pm 1}, \\
K_a E_{b\ b\pm 1} K_a^{-1} &= q^{(\epsilon_a,\, \epsilon_b - \epsilon_{b\pm 1})} E_{b\ b\pm 1}, \\
[E_{a\,a+1},\ E_{b+1\,b}\} &= \delta_{ab} \frac{K_a K_{a+1}^{-1} - K_a^{-1} K_{a+1}}{q_a - q_a^{-1}}, \\
(E_{m\,m+1})^2 &= (E_{m+1\,m})^2 = 0, \\
E_{a\,a+1} E_{b\,b+1} &= E_{b\,b+1} E_{a\,a+1}, \\
E_{a+1\,a} E_{b+1\,b} &= E_{b+1\,b} E_{a+1\,a}, \quad |a-b| \geq 2, \\
\mathcal{S}^{(+)}_{a\ a\pm 1} &= \mathcal{S}^{(-)}_{a\ a\pm 1} = 0, \quad a \neq m, \\
\{E_{m-1\,m+2},\ E_{m\,m+1}\} &= \{E_{m+2\,m-1},\ E_{m+1\,m}\} = 0,
\end{aligned} \tag{2.1}$$

where $[E_{a\,a+1},\, E_{b+1\,b}\} := E_{a\,a+1}\, E_{b+1\,b} - (-1)^{\delta_{am}\delta_{bm}} E_{b+1\,b}\, E_{a\,a+1}$. The $E_{m-1\,m+2}$ and $E_{m+2\,m-1}$ are the $a = m-1$, $b = m+1$, cases of the elements defined by (2.2), and

$$\begin{aligned}
\mathcal{S}^{(+)}_{a\ a\pm 1} &= (E_{a\,a+1})^2 E_{a\pm 1\,a+1\pm 1} - (q + q^{-1}) E_{a\,a+1}\ E_{a\pm 1\,a+1\pm 1}\ E_{a\,a+1} \\
&+ E_{a\pm 1\,a+1\pm 1}\ (E_{a\,a+1})^2, \\
\mathcal{S}^{(-)}_{a\ a\pm 1} &= (E_{a+1\,a})^2\, E_{a+1\pm 1\,a\pm 1} - (q + q^{-1}) E_{a+1\,a}\ E_{a+1\pm 1\,a\pm 1}\ E_{a+1\,a} \\
&+ E_{a+1\pm 1\,a\pm 1}\ (E_{a+1\,a})^2.
\end{aligned}$$

The $\mathbb{Z}_2$ grading of the superalgebra is defined by declaring the elements $K_a^{\pm 1}$, $\forall a \in \mathbf{I}$, and $E_{b\,b+1}$, $E_{b+1\,b}$, $b \neq m$, to be even and $E_{m\,m+1}$ and $E_{m+1\,m}$ to be odd. Throughout the paper, we use $[f]$ to denote the parity of the element f of any $\mathbb{Z}_2$-graded space.

It is well known that $\mathrm{U}_q(\mathfrak{g})$ has the structure of a Hopf superalgebra [13, 14], with a co - multiplication

$$\begin{aligned}
\Delta(E_{a\,a+1}) &= E_{a\,a+1} \otimes K_a K_{a+1}^{-1} + 1 \otimes E_{a\,a+1}, \\
\Delta(E_{a+1\,a}) &= E_{a+1\,a} \otimes 1 + K_a^{-1} K_{a+1} \otimes E_{a+1\,a}, \\
\Delta(K_a^{\pm 1}) &= K_a^{\pm 1} \otimes K_a^{\pm 1},
\end{aligned}$$

co - unit

$$\begin{aligned}
\epsilon(E_{a\,a+1}) &= E_{a+1\,a} = 0, \quad \forall a \in \mathbf{I}', \\
\epsilon(K_b^{\pm 1}) &= 1, \quad \forall b \in \mathbf{I},
\end{aligned}$$

and antipode

$$\begin{aligned}
S(E_{a\,a+1}) &= -E_{a\,a+1} K_a^{-1} K_{a+1}, \\
S(E_{a+1\,a}) &= -K_a K_{a+1}^{-1} E_{a+1\,a}, \\
S(K_a^{\pm 1}) &= K_a^{\mp 1} \otimes K_a^{\mp 1}.
\end{aligned}$$

Let E_{ab}, E_{ba}, $a < b$, be elements of $\mathrm{U}_q(\mathfrak{g})$ defined by

$$\begin{aligned}
E_{ab} &= E_{ac} E_{cb} - q_c^{-1} E_{cb} E_{ac}, \quad a < c < b, \\
E_{ba} &= E_{bc} E_{ca} - q_c E_{ca} E_{bc}, \quad a < c < b.
\end{aligned} \tag{2.2}$$

The definition is independent of the c chosen [20]. These elements are the generalization to $\mathrm{U}_q(\mathfrak{gl}_{m|n})$ of a similar set of elements for $U_q(\mathfrak{gl}_n)$ constructed by Jimbo [6]. They were used in [9] for the construction of the universal R-matrix of $\mathrm{U}_q(\mathfrak{gl}_{m|n})$. We mention that the E_{ab} behave very much like the images of e_{ab} in the universal enveloping algebra of $\mathfrak{gl}_{m|n}$. For example, $E_{ab}^2 = E_{ba}^2 = 0$, if $a \leq m < b$. More importantly, we have the following Poincaré-Birkhoff-Witt theorem for $\mathrm{U}_q(\mathfrak{g})$.

Theorem 2.1. [20, 23] *The ordered products of non-negative powers of all the E_{ab}, $a \neq b$, and integer powers of all K_c, $c \in \mathbf{I}$, with respect to any linear ordering of the elements of $\{E_{ab}, E_{ba} | a < b\} \cup \{K_c | c \in \mathbf{I}\}$ form a basis of $\mathrm{U}_q(\mathfrak{g})$.*

We shall assume that every $\mathrm{U}_q(\mathfrak{g})$-module to be considered in this paper is $\mathbb{Z}_2$-graded. Let V be a $\mathrm{U}_q(\mathfrak{g})$-module. A weight vector $v \in V$ is the simultaneous eigenvector of all the K_a, $a \in \mathbf{I}$. We shall say that v is an integral weight vector with weight $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$ if

$$K_a v = q^{(\mu, \epsilon_a)} v, \qquad \forall a.$$

We shall denote by L_λ, $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$, the irreducible $\mathrm{U}_q(\mathfrak{g})$-module with a highest weight vector which is an integral weight vector with weight λ. The λ will be referred to as the highest weight of L_λ. It was shown in [20] that L_λ is finite dimensional if and only if its highest weight is dominant.

We shall call a Hopf sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$ a quantum sub-superalgebra. Corresponding to a regular Lie sub-superalgebra $\mathfrak{r}$ of $\mathfrak{g}$ specified by the sets Θ_0 and $\Theta_\pm$, there exists an associated quantum sub-superalgebra $\mathrm{U}_q(\mathfrak{r})$ generated by the elements of the following set $\{K_a,\, K_a^{-1} | a \in \Theta_0\} \cup \{E_{b,b+1} | b \in \Theta_+\} \cup \{E_{c+1,c} | c \in \Theta_-\}$. Important quantum sub-superalgebras are $\mathrm{U}_q(\mathfrak{h})$ and the two quantum Borel sub-superalgebras $\mathrm{U}_q(\mathfrak{b})$ and $\mathrm{U}_q(\bar{\mathfrak{b}})$. If $\mathfrak{r}$ is a parabolic (respectively reductive) Lie sub-superalgebra of $\mathfrak{g}$, then $\mathrm{U}_q(\mathfrak{r})$ will be called a parabolic (respectively reductive) quantum sub-superalgebra

of $\mathrm{U}_q(\mathfrak{g})$. If $\mathrm{U}_q(\mathfrak{p})$ is parabolic with the Levi factor $\mathrm{U}_q(\mathfrak{l})$, then we have the Hopf superalgebra inclusions $\mathrm{U}_q(\mathfrak{l}) \subset \mathrm{U}_q(\mathfrak{p}) \subset \mathrm{U}_q(\mathfrak{g})$.

For later use, we consider here a particular module over a reductive quantum subsuperalgebra $\mathrm{U}_q(\mathfrak{l})$. Let $\mathrm{U_q}(\mathfrak{l}_0) = \mathrm{U}_q(\mathfrak{l}) \cap \mathrm{U_q}(\mathfrak{g}_0)$, and denote by $\mathrm{U_q}(\mathfrak{l}_{\leq 0})$ the Hopf subsuperalgebra of $\mathrm{U}_q(\mathfrak{l})$ generated by all the generators of $\mathrm{U}_q(\mathfrak{l})$ but $E_{m,m+1}$. Note that if $\mathrm{U}_q(\mathfrak{l}) \subset \mathrm{U}_q(\mathfrak{g}_0)$, then $\mathrm{U_q}(\mathfrak{l}_{\leq 0}) = \mathrm{U_q}(\mathfrak{l})$. Let $L_\mu^{(\mathfrak{l}_{\leq 0})}$ be the irreducible $\mathrm{U_q}(\mathfrak{l}_{\leq 0})$-module with integral $\mathrm{U}_q(\mathfrak{l}_0)$ highest weight μ. Note that the generator $E_{m+1,n}$ of $\mathrm{U_q}(\mathfrak{l}_{\leq 0})$ necessarily acts on $L_\mu^{(\mathfrak{l}_{\leq 0})}$ by zero. Furthermore, $L_\mu^{(\mathfrak{l}_{\leq 0})}$ restricts to an irreducible $\mathrm{U}_q(\mathfrak{l}_0)$-module.

Definition 2.1.

$$K_\mu^{(\mathfrak{l})} := \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{l}_{\leq 0})}\left(\mathrm{U}_q(\mathfrak{l}),\, L_\mu^{(\mathfrak{l}_{\leq 0})}\right). \tag{2.3}$$

This will be referred to as a dual Kac module over $\mathrm{U}_q(\mathfrak{l})$. The action of any $y \in \mathrm{U}_q(\mathfrak{l})$ on $\zeta \in K_\mu^{(\mathfrak{l})}$ is given by $\langle y\zeta,\, x\rangle = (-1)^{[y]([x]+[\zeta])}\langle \zeta,\, xy\rangle$, $\forall x \in \mathrm{U}_q(\mathfrak{l})$.

Let $Ker\epsilon_{\leq 0}$ be the subspace of $\mathrm{U}_q(\mathfrak{l}_{\leq 0})$ annihilated by the co-unit ϵ. It generates a two-sided ideal $J(\mathfrak{l})$ of $\mathrm{U}_q(\mathfrak{l})$. By using the PBW theorem 2.1 (generalized in the obvious way to $\mathrm{U}_q(\mathfrak{l})$) we can easily show that

$$K_\mu^{(\mathfrak{l})} \;=\; (\mathrm{U}_q(\mathfrak{l})/J(\mathfrak{l}))^* \otimes_{\mathbb{C}(q)} L_\mu^{(\mathfrak{l}_{\leq 0})}. \tag{2.4}$$

It again follows from the PBW theorem that $\mathrm{U}_q(\mathfrak{l})/J(\mathfrak{l})$ is finite dimensional. The $\mathrm{U}_q(\mathfrak{h})$-module structure of its dual vector space can be described explicitly. Let $\Phi_1^+(\mathfrak{l})$ be the set of the odd positive roots of $\mathfrak{l}$. Let E be the $\mathrm{U}_q(\mathfrak{h})$-module with a basis $\{v_\gamma | \gamma \in \Phi_1^+(\mathfrak{l})\}$ such that $K_a v_\gamma = q^{-(\gamma,\epsilon_a)} v_\gamma$, for all a. The exterior algebra of E forms a $\mathrm{U}_q(\mathfrak{h})$-module, which we denote by $\Lambda(\mathfrak{l}_{-1})$. Then $(\mathrm{U}_q(\mathfrak{l})/J(\mathfrak{l}))^*$ is isomorphic to $\Lambda(\mathfrak{l}_{-1})$. Therefore, as a $\mathrm{U}_q(\mathfrak{h})$-module (with diagonal action),

$$K_\mu^{(\mathfrak{l})} \;\cong\; \Lambda(\mathfrak{l}_{-1}) \otimes_{\mathbb{C}(q)} L_\mu^{(\mathfrak{l}_{\leq 0})}. \tag{2.5}$$

Note that when $\mathrm{U}_q(\mathfrak{l})$ is purely even, $K_\mu^{(\mathfrak{l})} = L_\mu^{(\mathfrak{l}_{\leq 0})}$.

3. Quantum homogeneous Super Vector Bundles

In this section we introduce quantum homogeneous superspaces and quantum homogeneous supervector bundles in the context of the quantum general linear supergroup. Let us begin be considering the Hopf superalgebra of functions on the quantum general linear supergroup.

3.1. **Functions on the quantum general linear supergroup.** General references on Hopf (super)algebra are [13, 14]. A treatment of the classical general linear supergroup similar to what to be presented here is given in [18]. Let $\mathrm{U}_q(\mathfrak{g})^*$ denote the $\mathbb{Z}_2$-graded dual vector space of $\mathrm{U}_q(\mathfrak{g})$. It has a natural associative superalgebraic structure induced by the co-superalgebraic structure of $\mathrm{U}_q(\mathfrak{g})$. Denote the multiplication of $\mathrm{U}_q(\mathfrak{g})^*$ by $m_\circ$, then $\langle m_\circ(f \otimes g),\, x\rangle = \langle f \otimes g,\, \Delta(x)\rangle$, for all $x \in \mathrm{U}_q(\mathfrak{g})$. (We shall use the notations $\phi(v)$ and $\langle \phi,\, x\rangle$ interchangeably for the image of $v \in V$ under $\phi \in \mathrm{Hom}_{\mathbb{C}(q)}(V, W)$.)

There exist two gradation preserving $\mathrm{U}_q(\mathfrak{g})$-actions on $\mathrm{U}_q(\mathfrak{g})^*$,

$$d\tilde{L} : \mathrm{U}_q(\mathfrak{g}) \otimes \mathrm{U}_q(\mathfrak{g})^* \to \mathrm{U}_q(\mathfrak{g})^*, \qquad x \otimes f \mapsto d\tilde{L}_x(f),$$
$$d\tilde{R} : \mathrm{U}_q(\mathfrak{g}) \otimes \mathrm{U}_q(\mathfrak{g})^* \to \mathrm{U}_q(\mathfrak{g})^*, \qquad x \otimes f \mapsto d\tilde{R}_x(f),$$

defined by

$$\begin{aligned}\langle d\tilde{L}_x(f),\, y\rangle &= (-1)^{[x][f]}\langle f,\, S(x)y\rangle,\\ \langle d\tilde{R}_x(f),\, y\rangle &= (-1)^{[x]}\langle f,\, yx\rangle, \quad \forall y\in \mathrm{U}_q(\mathfrak{g}).\end{aligned}$$

It is easy to see that $d\tilde{L}_{xy} = d\tilde{L}_x d\tilde{L}_y$, and $d\tilde{R}_{xy} = d\tilde{R}_x d\tilde{R}_y$, for all $x, y \in \mathrm{U}_q(\mathfrak{g})$. Straightforward calculations show that each of these actions converts $\mathrm{U}_q(\mathfrak{g})^*$ into a (graded) left $\mathrm{U}_q(\mathfrak{g})$-module. Furthermore, with respect to the module structure the product map of $\mathrm{U}_q(\mathfrak{g})^*$ is a $\mathrm{U}_q(\mathfrak{g})$-module homomorphism and the unit element of $\mathrm{U}_q(\mathfrak{g})^*$ is $\mathrm{U}_q(\mathfrak{g})$-invariant. Therefore, each of these actions converts $\mathrm{U}_q(\mathfrak{g})^*$ into a left $\mathrm{U}_q(\mathfrak{g})$-module superalgebra. The fact that the product map in $\mathrm{U}_q(\mathfrak{g})^*$ is a module homomorphism means that the operators $d\tilde{R}_x$ and $d\tilde{L}_x$ behave as some sort of generalized super derivations. Indeed, for all $f,\, g\in \mathrm{U}_q(\mathfrak{g})^*$, we have

$$d\tilde{R}_x(fg) = \sum_{(x)} (-1)^{[x_{(2)}][f]} d\tilde{R}_{x_{(1)}}(f) d\tilde{R}_{x_{(2)}}(g), \tag{3.1}$$

where we have used the standard Sweedler notation for the co-multiplication of x. However, for $d\tilde{L}$, we have

$$d\tilde{L}_x(fg) = \sum_{(x)} (-1)^{[x'_{(2)}][f]} d\tilde{L}_{x'_{(1)}}(f) d\tilde{L}_{x'_{(2)}}(g), \tag{3.2}$$

with respect to the opposite co-multiplication $\Delta'(x) = \sum_{(x)} x'_{(1)} \otimes x'_{(2)}$ of x. The two actions also super-commute in the sense that $d\tilde{L}_x d\tilde{R}_y = (-1)^{[x][y]} d\tilde{R}_y d\tilde{L}_x$, for all $x, y \in \mathrm{U}_q(\mathfrak{g})$.

Let $\mathrm{U}_q(\mathfrak{g})^\circ := \{f\in \mathrm{U}_q(\mathfrak{g})^* \mid \text{kernel of } f \text{ contains a co-finite ideal of } \mathrm{U}_q(\mathfrak{g})\}$ be the finite dual [14] of the quantized universal enveloping algebra $\mathrm{U}_q(\mathfrak{g})$ of $\mathfrak{g}$. Here we remark again that we only consider $\mathbb{Z}_2$-graded subalgebras and (left, right or two-sided) ideals of $\mathrm{U}_q(\mathfrak{g})$ in this paper. We have the following lemma, which is an adaption of a standard result (see, e.g., Lemma 9.1.1 in [14]) on ordinary associative algebras to $\mathrm{U}_q(\mathfrak{g})$. In fact the result is valid for any associative superalgebra.

Lemma 3.1. *For any $f\in \mathrm{U}_q(\mathfrak{g})^*$, the following conditions are equivalent:*

(1) *f vanishes on a left ideal of $\mathrm{U}_q(\mathfrak{g})$ of finite co-dimension;*
(2) *f vanishes on a right ideal of $\mathrm{U}_q(\mathfrak{g})$ of finite co-dimension;*
(3) *f vanishes on an ideal of $\mathrm{U}_q(\mathfrak{g})$ of finite co-dimension, thus belongs to $\mathrm{U}_q(\mathfrak{g})^\circ$;*
(4) *$d\tilde{L}_{\mathrm{U}_q(\mathfrak{g})}(f)$ is finite dimensional;*
(5) *$d\tilde{R}_{\mathrm{U}_q(\mathfrak{g})}(f)$ is finite dimensional;*
(6) *$\left(d\tilde{L}_{\mathrm{U}_q(\mathfrak{g})} \otimes d\tilde{R}_{\mathrm{U}_q(\mathfrak{g})}\right)(f)$ is finite dimensional;*
(7) *$m^*(f)\in \mathrm{U}_q(\mathfrak{g})^*\otimes \mathrm{U}_q(\mathfrak{g})^*$, where $m^* : \mathrm{U}_q(\mathfrak{g})^* \to (\mathrm{U}_q(\mathfrak{g})\otimes \mathrm{U}_q(\mathfrak{g}))^*$ is defined by $\langle m^*(f),\, x\otimes y\rangle = \langle f,\, xy\rangle$ for all $x, y\in \mathrm{U}_q(\mathfrak{g})$.*

Proof. The proof of Lemma 9.1.1 in [14] can be extended verbatim to superalgebras. □

Therefore, f belongs to $\mathrm{U}_q(\mathfrak{g})^\circ$ if and only if one of the equivalent conditions are satisfied. The Lemma in particular enables us to impose a Hopf superalgebra structure on $\mathrm{U}_q(\mathfrak{g})^\circ$, with multiplication $m_\circ$, co-multiplication $\Delta_\circ = m^*|_{\mathrm{U}_q(\mathfrak{g})^\circ}$, unit being ϵ, and co-unit being the unit $\mathbb{1}_{\mathrm{U}_q(\mathfrak{g})}$ of $\mathrm{U}_q(\mathfrak{g})$. The antipode $S_\circ$ of $\mathrm{U}_q(\mathfrak{g})^\circ$ is defined by

$$\langle S_\circ(f),\, x\rangle = \langle f,\, S(x)\rangle, \quad \forall f\in \mathrm{U}_q(\mathfrak{g})^\circ,\ x\in \mathrm{U}_q(\mathfrak{g}).$$

Recall that the antipode $S_\circ$ is invertible since S is. For convenience, we shall drop the subscript $\circ$ from the notations for all the structure maps but the antipode of $\mathrm{U}_q(\mathfrak{g})^\circ$.

A $\mathrm{U}_q(\mathfrak{g})$-representation π in $d\times d$-matrices is a superalgebraic map which is homogeneous of degree 0. We write $\pi(x) = (\pi_{ij}(x))_{i,j=1}^d$ for any $x \in \mathrm{U}_q(\mathfrak{g})$. Define $\pi_{ij} \in \mathrm{U}_q(\mathfrak{g})^*$, $i, j = 1, 2, \ldots, d$, by

$$\langle \pi_{ij},\, x\rangle := \pi_{ij}(x), \qquad \forall i, j,$$

and call them the matrix elements of π. Since the kernel of any finite dimensional representation is an ideal of $\mathrm{U}_q(\mathfrak{g})$ with finite co-dimension, all the matrix elements of the representation belong to $\mathrm{U}_q(\mathfrak{g})^\circ$. Conversely, $\mathrm{U}_q(\mathfrak{g})^\circ$ is spanned by the matrix elements of all the finite dimensional representations of $\mathrm{U}_q(\mathfrak{g})$. To see this, we only need to consider an arbitrary non-zero element $f \in \mathrm{U}_q(\mathfrak{g})^\circ$. Let K be a graded co-finite ideal of $\mathrm{U}_q(\mathfrak{g})$ contained in the kernel of f. Then $U(\mathfrak{g})/K$ forms a left $\mathrm{U}_q(\mathfrak{g})$-module under left multiplication,

$$\begin{aligned} \mathrm{U}_q(\mathfrak{g}) \otimes U(\mathfrak{g})/K &\ \to\ U(\mathfrak{g})/K, \\ y \otimes (x + K) &\ \mapsto\ yx + K. \end{aligned}$$

Let $\{x_i + K\}$ be a basis of $\mathrm{U}_q(\mathfrak{g})/K$, and denote by f_{ij} the matrix elements of the associated representation relative to this basis. Choose $c_i \in \mathbb{C}(q)$ such that $\mathbb{1}_{\mathrm{U}_q(\mathfrak{g})} + K = \sum_i c_i x_i + K$, where $\mathbb{1}_{\mathrm{U}_q(\mathfrak{g})} + K$ is not contained in the kernel of f as a set since $f \neq 0$. Then $f = \sum_{i,j} c_i \langle f,\, x_j\rangle f_{ji}$.

Definition 3.1. *Let $\mathcal{A}(\mathfrak{g})$ be the $\mathbb{Z}_2$-graded subspace of $\mathrm{U}_q(\mathfrak{g})^\circ$ spanned by the matrix elements of the $\mathrm{U}_q(\mathfrak{g})$-representations furnished by finite dimensional objects of $\mathcal{C}(\mathfrak{g}, \mathfrak{g})$.*

Lemma 3.2. *$\mathcal{A}(\mathfrak{g})$ is a Hopf sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})^\circ$.*

Proof. The space spanned by the matrix elements of any finite dimensional representation of $\mathrm{U}_q(\mathfrak{g})$ is a sub co-algebra of $\mathrm{U}_q(\mathfrak{g})^\circ$. Thus $\mathcal{A}(\mathfrak{g})$ forms a sub co-algebra of $\mathrm{U}_q(\mathfrak{g})^\circ$. Since $\mathcal{C}(\mathfrak{g}, \mathfrak{g})$ is closed under tensor product, $\mathcal{A}(\mathfrak{g})$ is closed under multiplication. Also, if a finite dimensional $\mathrm{U}_q(\mathfrak{g})$-module belongs to $\mathcal{C}(\mathfrak{g}, \mathfrak{g})$, then so is also its dual. Thus $\mathcal{A}(\mathfrak{g})$ is stable under the antipode of $\mathrm{U}_q(\mathfrak{g})^\circ$. $\square$

Remark 3.1. *From the discussion on matrix elements of finite dimensional representations we see that $f \in \mathrm{U}_q(\mathfrak{g})^\circ$ belongs to $\mathcal{A}(\mathfrak{g})$ if it satisfies either $d\tilde{L}_{K_a}(f) = q^{(\mu,\epsilon_a)} f$, $\forall a \in \mathbf{I}$, or $d\tilde{R}_{K_a}(f) = q^{(\mu,\epsilon_a)} f$, $\forall a \in \mathbf{I}$, for some $\mu \in \mathfrak{h}^*_{\mathbb{Z}}$.*

3.2. **Quantum homogeneous super vector bundles.** Recall that in classical geometry, a compact manifold can be recovered from its algebra of continuous functions by the Gelfand-Naimark theorem. Also, the Serre-Swan theorem establishes a one to one correspondence between the spaces of the continuous sections of vector bundles over a compact manifold and the finite type projective modules of the algebra of continuous functions on the manifold. These results are taken as the starting point for non-commutative geometry [3], where 'manifolds' are replaced by non-commutative algebras, and 'vector bundles' by finitely generated projective modules. The quantum homogeneous superspaces and quantum homogeneous supervector bundles to be studied here are defined in this spirit.

As is well known, all holomorphic functions on a compact complex manifold are constants. Therefore, the algebra of holomorphic functions contains little information about the manifold itself. This problem persists in classical supergeometry [10] and also quantum geometry [5]. However, as shown in [5] in the context of ordinary quantum

groups, we can get around the problem by working with the quantum analogues of smooth functions in a real setting. To do this, we need some basic notions about $*$-Hopf superalgebras [21, 22].

A $*$-superalgebraic structure on an associative superalgebra A over $\mathbb{C}(q)$ is a conjugate linear anti-involution $\theta : A \to A$: for all $x, y \in A$, $c, c' \in \mathbb{C}(q)$,

$$\theta(cx + dy) = \bar{c}\theta(x) + \bar{c}'\theta(y), \qquad \theta(xy) = \theta(y)\theta(x), \qquad \theta^2(x) = x. \tag{3.3}$$

Here $\bar{c}$ and $\bar{c}'$ are defined in the following way. Let P be a complex polynomial in q. Then $\bar{P}$ is the polynomial obtained by replacing all the coefficients of P by their complex conjugates. Now if there is another polynomial Q in q such that $c = P/Q$, then $\bar{c} = \bar{P}/\bar{Q}$.

Remark 3.2. *Note that the second equation in* (3.3) *does not involve any sign factors as one would normally expect of superalgebras.*

We shall sometimes use the notation (A, θ) for the $*$-superalgebra A with the $*$-structure θ. Let (B, θ_1) be another associative $*$-superalgebra. Now $A \otimes B$ has a natural superalgebra structure, with the multiplication defined for any $a, a' \in A$ and $b, b' \in B$ by

$$(a \otimes b)(a' \otimes b') \;\; = \;\; (-1)^{[b][a']} aa' \otimes bb'.$$

Furthermore, the following conjugate linear map

$$\theta \star \theta_1 : a \otimes b \mapsto (1 \otimes \theta_1(b))(\theta(a) \otimes 1) = (-1)^{[a][b]}\theta(a) \otimes \theta_1(b) \tag{3.4}$$

defines a $*$-superalgebraic structure on $A \otimes B$.

Let us assume that A is a Hopf superalgebra with co-multiplication Δ, co-unit ϵ and antipode S. If the $*$-superalgebraic structure θ satisfies

$$(\theta \star \theta)\Delta = \Delta\theta, \qquad \theta\epsilon = \epsilon\theta,$$

then A is called a Hopf $*$-superalgebra. Now

$$\sigma := S\theta$$

satisfies $\sigma^2 = id_A$, as follows from the definition of the antipode.

Let A^0 denote the finite dual of A, which has a natural Hopf superalgebraic structure. If A is a Hopf $*$-superalgebra with the $*$-structure θ, then $\sigma = S\theta$ induces a map $\omega : A^0 \to A^0$ defined for any $f \in A^0$ by

$$\langle \omega(f), x \rangle = \overline{\langle f, \sigma(x) \rangle}, \quad \forall x \in A. \tag{3.5}$$

As can be easily shown [22], this map ω gives rise to a Hopf $*$-superalgebraic structure on A^0.

In the case of $\mathrm{U}_q(\mathfrak{g})$, the following conjugate anti-involution defines a Hopf $*$-superalgebra structure:

$$\begin{aligned} \theta: \quad E_{a,a+1} &\mapsto E_{a+1,a}K_a K_{a+1}^{-1}, \\ E_{a+1,a} &\mapsto K_a^{-1}K_{a+1}E_{a,a+1}, \quad \forall a \in \mathbf{I}', \\ K_b &\mapsto K_b, \quad \forall b \in \mathbf{I}. \end{aligned}$$

The classical counter part of this map determines a compact real form of the complex general linear supergroup. Let $\sigma = S\theta$, and define

$$\mathrm{U}_q^{\mathbb{R}}(\mathfrak{g}) \;\; := \;\; \{x \in \mathrm{U}_q(\mathfrak{g}) | \sigma(x) = x\}. \tag{3.6}$$

Clearly $\mathrm{U}_q^{\mathbb{R}}(\mathfrak{g})$ forms an associative superalgebra over $\mathbb{R}(q)$, even though it may not have a Hopf superalgebra structure. We shall refer to it as a real form of $\mathrm{U}_q(\mathfrak{g})$, as $\mathrm{U}_q(\mathfrak{g}) = \mathbb{C}(q) \otimes_{\mathbb{R}(q)} \mathrm{U}_q^{\mathbb{R}}(\mathfrak{g})$.

Now $\mathrm{U}_q(\mathfrak{g})^\circ$ acquires a Hopf $*$-superalgebra structure ω which is induced by σ. It is easy to show that the image under ω of any matrix element of a finite dimensional $\mathrm{U}_q(\mathfrak{g})$-representation with integral weights must again be a matrix element of a $\mathrm{U}_q(\mathfrak{g})$-representation with the same properties. Thus

Lemma 3.3. *$\mathcal{A}(\mathfrak{g})$ forms a Hopf $*$-superalgebra.*

Therefore $\mathcal{A}(\mathfrak{g})$ should be considered as some 'complexification' of the superalgebra of functions on a compact form of the quantum general linear supergroup.

Let us denote by dL_x and dR_x respectively the restrictions of $d\tilde{L}_x$ and $d\tilde{R}_x$ to $\mathcal{A}(\mathfrak{g})$. The following definition will be important for the remainder of the paper. Let $\mathrm{U}_q(\mathfrak{l})$ be a reductive quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$. Set $\mathrm{U}_q^{\mathbb{R}}(\mathfrak{l}) := \mathrm{U}_q(\mathfrak{l}) \cap \mathrm{U}_q^{\mathbb{R}}(\mathfrak{g})$. We now consider the sub-superalgebra of $\mathcal{A}(\mathfrak{g})$ invariant under the left translation of $\mathrm{U}_q^{\mathbb{R}}(\mathfrak{l})$.

Definition 3.2. *Define*

$$\mathcal{A}(\mathfrak{g}, \mathfrak{l}) := \left\{ f \in \mathcal{A}(\mathfrak{g}) \mid dL_x(f) = \epsilon(x) f, \ \forall x \in \mathrm{U}_q^{\mathbb{R}}(\mathfrak{l}) \right\}, \tag{3.7}$$

where ϵ is the co-unit of $\mathrm{U}_q(\mathfrak{g})$.

We shall show presently that $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$ forms a superalgebra. In the philosophy of non-commutative geometry [3], $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$ defines some virtue quantum homogeneous superspace.

It is useful to compare the situation with classical supergeometry [10]. Let Λ be a finite dimensional Grassmann algebra. Take a parabolic subgroup P of $GL(m|n;\Lambda)$ with Lie superalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ where $\mathfrak{u}$ is some nilpotent ideal of $\mathfrak{p}$. Let $U(m|n)$ be a compact real form of $GL(m|n;\Lambda)$, and set $K = P \cap U(m|n)$. Then we have the symmetric superspace $U(m|n)/K$. The tensor product of Λ with the classical analogue of $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$ should capture the essential information of the complexification of the superalgebra of functions on $U(m|n)/K$.

Since $\mathbb{C}(q) \otimes_{\mathbb{R}(q)} \mathrm{U}_q^{\mathbb{R}}(\mathfrak{l}) = \mathrm{U}_q(\mathfrak{l})$, we can show that an element f of $\mathcal{A}(\mathfrak{g})$ belongs to $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$ if and only if

$$dL_x(f) = \epsilon(x) f, \quad \forall x \in \mathrm{U}_q(\mathfrak{l}).$$

Also, by Remark 3.1, an element g of $\mathrm{U}_q(\mathfrak{g})^\circ$ belongs to $\mathcal{A}(\mathfrak{g})$ if $dL_k(f) = \epsilon(k) f$, $\forall k \in \mathrm{U}_q(\mathfrak{h})$. Combining these observations, we arrive at the following result.

Lemma 3.4.

$$\mathcal{A}(\mathfrak{g}, \mathfrak{l}) \ = \ \{ f \in \mathrm{U}_q(\mathfrak{g})^\circ \mid dL_x(f) = \epsilon(x) f, \ \forall x \in \mathrm{U}_q(\mathfrak{l}) \}.$$

Using the left $\mathrm{U}_q(\mathfrak{g})$-module algebra structure of $\mathcal{A}(\mathfrak{g})$, we immediately show that

Lemma 3.5. *The $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$ is a sub-superalgebra of* $\mathcal{A}(\mathfrak{g})$.

Proof. $\mathrm{U}_q(\mathfrak{l})$, being a Hopf sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$, satisfies $\Delta(\mathrm{U}_q(\mathfrak{l})) \subset \mathrm{U}_q(\mathfrak{l}) \otimes \mathrm{U}_q(\mathfrak{l})$. If $f, \ g \in \mathcal{A}(\mathfrak{g}, \mathfrak{l})$, then by (3.2) we have $dL_x(fg) = \epsilon(x) fg$, $\forall x \in \mathrm{U}_q(\mathfrak{l})$. Therefore, $fg \in \mathcal{A}(\mathfrak{g}, \mathfrak{l})$. □

Since $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$ is non-commutative, there is a distinction between left and right $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-modules. However, the two sides of the story are 'mirror images' of each other, thus we shall consider $\mathbb{Z}_2$-graded left $\mathcal{A}(\mathfrak{g}, \mathfrak{l})$-modules only. A finitely generated projective

module over the superalgebra $\mathcal{A}(\mathfrak{g},\mathfrak{l})$ will be regarded as the space of sections of a quantum supervector bundle over the quantum homogeneous superspace.

Definition 3.3. *Let Ξ be a finite dimensional $\mathrm{U}_q(\mathfrak{l})$-module, which naturally restricts to a $\mathrm{U}_q^{\mathbb{R}}(\mathfrak{l})$-module. Define*

$$\mathcal{S}(\Xi) \ := \ \left\{\zeta\in\Xi\otimes\mathcal{A}(\mathfrak{g}) \mid (\mathrm{id}\otimes dL_x)\zeta=(S(x)\otimes\mathrm{id})\zeta,\ \forall x\in\mathrm{U}_q^{\mathbb{R}}(\mathfrak{l})\right\}.$$

Again it can be easily shown that

$$\mathcal{S}(\Xi) \ = \ \{\zeta\in\Xi\otimes\mathcal{A}(\mathfrak{g}) \mid (\mathrm{id}\otimes dL_x)\zeta=(S(x)\otimes\mathrm{id})\zeta,\ \forall x\in\mathrm{U}_q(\mathfrak{l})\}. \tag{3.8}$$

This fact will be used to prove the following result.

Proposition 3.1. (1) *$\mathcal{S}(\Xi)$ forms a left $\mathcal{A}(\mathfrak{g},\mathfrak{l})$-module under the action*

$$\mathcal{A}(\mathfrak{g},\mathfrak{l})\otimes\mathcal{S}(\Xi)\to\mathcal{S}(\Xi), \qquad f\otimes\zeta\mapsto f\zeta, \tag{3.9}$$

defined by $f\zeta:=\sum(-1)^{[f][w_i]}w_i\otimes fa_i$ for $\zeta=\sum w_i\otimes a_i$.

(2) *Every $\mathrm{U}_q(\mathfrak{l})$-module map $\phi:\Xi\to\Xi'$ induces an $\mathcal{A}(\mathfrak{g},\mathfrak{l})$-module homomorphism*

$$\mathcal{S}(\phi)=\phi\otimes\mathrm{id}:\mathcal{S}(\Xi)\to\mathcal{S}(\Xi'). \tag{3.10}$$

Proof. For $f\in\mathcal{A}(\mathfrak{g},\mathfrak{l})$ and $\zeta=\sum w_i\otimes a_i\in\mathcal{S}(\Xi)$, we have

$$\begin{aligned}(\mathrm{id}\otimes dL_x)f\zeta &= \sum(-1)^{([f]+[x])[w_i]}w_i\otimes dL_x(fa_i)\\ &= \sum(-1)^{([f]+[x])[w_i]+[x][f]}w_i\otimes f\,dL_x(a_i)\\ &= (S(x)\otimes\mathrm{id})f\zeta, \quad \forall x\in\mathrm{U}_q(\mathfrak{l}),\end{aligned}$$

where the second step uses (3.2) and the defining property of $\mathcal{A}(\mathfrak{g},\mathfrak{l})$, while the third uses the definition of $\mathcal{S}(\Xi)$. This proves the first claim.

The second claim is quite obvious. □

A quantum homogeneous supervector bundle is called trivial if it is isomorphic to a free left $\mathcal{A}(\mathfrak{g},\mathfrak{l})$-module. Quantum homogeneous supervector bundles induced by $\mathrm{U}_q(\mathfrak{g})$-modules are all trivial.

Proposition 3.2. *$\mathcal{S}(\Xi)$ is freely generated over $\mathcal{A}(\mathfrak{g},\mathfrak{l})$ if Ξ is the restriction of a finite dimensional $\mathrm{U}_q(\mathfrak{g})$-module.*

Proof. The proof is adapted from [5]. Note that a finite dimensional left $\mathrm{U}_q(\mathfrak{g})$-module Ξ has a natural right $\mathrm{U}_q(\mathfrak{g})^\circ$-comodule structure

$$\delta:\Xi\to\Xi\otimes\mathrm{U}_q(\mathfrak{g})^\circ, \quad \delta(v)(x)=(-1)^{[v][x]}xv, \quad \forall v\in\Xi,\ x\in\mathrm{U}_q(\mathfrak{g}).$$

Let $p:\mathrm{U}_q(\mathfrak{g})^\circ\otimes\mathrm{U}_q(\mathfrak{g})^\circ\to\mathrm{U}_q(\mathfrak{g})^\circ\otimes\mathrm{U}_q(\mathfrak{g})^\circ$ be defined by $f\otimes g\mapsto(-1)^{[f][g]}g\otimes f$. Define a map $\kappa:\Xi\otimes\mathrm{U}_q(\mathfrak{g})^\circ\to\Xi\otimes\mathrm{U}_q(\mathfrak{g})^\circ$ by the composition of the following maps

$$\Xi\otimes\mathrm{U}_q(\mathfrak{g})^\circ\xrightarrow{\delta\otimes\mathrm{id}}\Xi\otimes\mathrm{U}_q(\mathfrak{g})^\circ\otimes\mathrm{U}_q(\mathfrak{g})^\circ\xrightarrow{\mathrm{id}\otimes p(S^{-1}\otimes\mathrm{id})}\Xi\otimes\mathrm{U}_q(\mathfrak{g})^\circ\otimes\mathrm{U}_q(\mathfrak{g})^\circ\xrightarrow{\mathrm{id}\otimes m_o}\Xi\otimes\mathrm{U}_q(\mathfrak{g})^\circ,$$

where m_o is the multiplication of $\mathrm{U}_q(\mathfrak{g})^\circ$. Explicitly,

$$\begin{aligned}\zeta &= \sum v^{(i)}\otimes f^{(i)}\in\Xi\otimes\mathrm{U}_q(\mathfrak{g})^\circ,\\ \kappa(\zeta) &= \sum(-1)^{[f^{(i)}][v^{(i)}_{(2)}]}v^{(i)}_{(1)}\otimes f^{(i)}S^{-1}(v^{(i)}_{(2)}),\end{aligned} \tag{3.11}$$

where we have used Sweedler's notation for $\delta(v^{(i)})$. The inverse of κ is given by the composition of the following maps

$$\Xi\otimes\mathrm{U}_q(\mathfrak{g})^\circ\xrightarrow{\delta\otimes\mathrm{id}}\Xi\otimes\mathrm{U}_q(\mathfrak{g})^\circ\otimes\mathrm{U}_q(\mathfrak{g})^\circ\xrightarrow{\mathrm{id}\otimes p}\Xi\otimes\mathrm{U}_q(\mathfrak{g})^\circ\otimes\mathrm{U}_q(\mathfrak{g})^\circ\xrightarrow{\mathrm{id}\otimes m_o}\Xi\otimes\mathrm{U}_q(\mathfrak{g})^\circ.$$

The restriction of κ to $\mathcal{S}(\Xi)$ is a left $\mathcal{A}(\mathfrak{g},\mathfrak{l})$-module map as can be easily seen from (3.11). By using (3.11), we can also show by a direct calculation that for any $\zeta\in\mathcal{S}(\Xi)$,

$$(\mathrm{id}\otimes dL_u)\kappa(\zeta) \;=\; \epsilon(u)\zeta, \quad \forall u\in \mathrm{U}_q(\mathfrak{l}),$$

that is $\kappa(\zeta)\in\Xi\otimes\mathcal{A}(\mathfrak{g},\mathfrak{l})$ by Lemma 3.4. Since κ is invertible, $\kappa\left(\mathcal{S}(\Xi)\right)=\Xi\otimes\mathcal{A}(\mathfrak{g},\mathfrak{l})$. $\square$

As an immediate consequence of the proposition, we obtain the following sufficient condition which renders $\mathcal{S}(\Xi)$ projective over $\mathcal{A}(\mathfrak{g},\mathfrak{l})$.

Corollary 3.1. *The $\mathcal{S}(\Xi)$ is projective over $\mathcal{A}(\mathfrak{g},\mathfrak{l})$ if there exists a $\mathrm{U}_q(\mathfrak{l})$-module $\Xi^\perp$ and a finite dimensional $\mathrm{U}_q(\mathfrak{g})$-module V such that $\Xi\oplus\Xi^\perp$ is isomorphic to the restriction of V to a $\mathrm{U}_q(\mathfrak{l})$-module.*

If $\mathrm{U}_q(\mathfrak{g})$ was an ordinary quantized universal enveloping algebra associated with a finite dimensional semi-simple Lie algebra, it was shown in [5] that $\mathcal{S}(\Xi)$ was always projective over $\mathcal{A}(\mathfrak{g},\mathfrak{l})$. Unfortunately this is no longer true for $\mathrm{U}_q(\mathfrak{gl}_{m|n})$. However, if $\mathrm{U}_q(\mathfrak{l})$ is a purely even reductive quantum subalgebra of $\mathrm{U}_q(\mathfrak{g})$, that is, $\mathrm{U}_q(\mathfrak{l})\subset\mathrm{U}_\mathrm{q}(\mathfrak{g}_0)$, then $\mathcal{S}(\Xi)$ is a finitely generated projective $\mathcal{A}(\mathfrak{g},\mathfrak{l})$-module. More generally, we have the following result.

Lemma 3.6. *If $\lambda\in\mathfrak{h}^*_{\mathbb{Z}}$ is $\mathfrak{g}$-dominant, then $\mathcal{S}(K^{(\mathfrak{l})}_\lambda)$ is projective over $\mathcal{A}(\mathfrak{g},\mathfrak{l})$, where $K^{(\mathfrak{l})}_\lambda$ is the dual Kac module over $\mathrm{U}_q(\mathfrak{l})$ defined by (2.3).*

Proof. To prove this, we let $\bar{L}^{(\mathfrak{g}_{\le 0})}_{-\lambda}$ be the irreducible $\mathrm{U}_q(\mathfrak{g}_{\le 0})$-module with lowest weight $-\lambda$, which is finite dimensional. Let $\bar{V}_{-\lambda}=\mathrm{U}_q(\mathfrak{g})\otimes_{\mathrm{U}_q(\mathfrak{g}_{\le 0})}\bar{L}^{(\mathfrak{g}_{\le 0})}_{-\lambda}$. Then $\bar{V}_{-\lambda}$ is a finite dimensional $\mathrm{U}_q(\mathfrak{g})$-module, which naturally restricts to a $\mathrm{U}_q(\mathfrak{l})$-module. Let $\bar{v}_{-\lambda}\in\bar{V}_{-\lambda}$ be a non-zero vector with weight $-\lambda$, which generates a $\mathrm{U}_q(\mathfrak{l})$-module $\bar{K}=\mathrm{U}_q(\mathfrak{l})\bar{v}_{-\lambda}$. Regard $\bar{V}_{-\lambda}$ as a $\mathrm{U}_q(\mathfrak{h})$-module, we have the decomposition $\bar{V}_{-\lambda}=\bar{K}\oplus\bar{K}^\perp$. This in fact is also a direct sum of $\mathrm{U}_q(\mathfrak{l})$-modules as the weights of $\bar{K}^\perp$ differ from those of $\bar{K}$ by roots not belonging to $\mathfrak{l}$. The dual $\bar{V}^*_{-\lambda}$ of $\bar{V}_{-\lambda}$ has a natural $\mathrm{U}_q(\mathfrak{l})$-module structure, and contains the $\mathrm{U}_q(\mathfrak{l})$-submodule $\bar{K}^*=K^{(\mathfrak{l})}_\lambda$ as a direct summand. Therefore Proposition 3.1 applies to the present situation. $\square$

The space $\mathcal{S}(\Xi)$ has a direct bearing on the representation theory of $\mathrm{U}_q(\mathfrak{g})$.

Lemma 3.7. (1) *$\mathcal{S}(\Xi)$ forms a $\mathrm{U}_q(\mathfrak{g})$-module under the action*

$$\mathrm{U}_q(\mathfrak{g})\otimes\mathcal{S}(\Xi)\to\mathcal{S}(\Xi), \qquad x\otimes\zeta\mapsto(\mathrm{id}\otimes dR_x)\zeta. \tag{3.12}$$

(2) *for every $\mathrm{U}_q(\mathfrak{l})$-module map $\phi:\Xi\to\Xi'$, the induced map*

$$\mathcal{S}(\phi)=\phi\otimes\mathrm{id}:\mathcal{S}(\Xi)\to\mathcal{S}(\Xi'), \tag{3.13}$$

is $\mathrm{U}_q(\mathfrak{g})$-equivariant.

Proof. The first part follows from the fact that the two actions dL and dR of $\mathrm{U}_q(\mathfrak{g})$ on $\mathcal{A}(\mathfrak{g})$ super-commute. To see the second part, let $\zeta=\sum v_i\otimes f_i$ be in $\mathcal{S}(\Xi)$. Then for all $x\in\mathrm{U}_q(\mathfrak{g})$,

$$\begin{aligned} x\circ(\mathcal{S}(\phi)\zeta) &= \sum(-1)^{([v_i]+[\phi])[x]}\phi(v_i)\otimes dR_x(f_i)\\ &= (-1)^{[x][\phi]}\mathcal{S}(\phi)(x\circ\zeta). \end{aligned}$$

$\square$

Of particular interest to us is the case when Ξ is a finite dimensional $\mathrm{U}_q(\mathfrak{p})$-module, where $\mathrm{U}_q(\mathfrak{p})$ is a parabolic quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$ with $\mathrm{U}_q(\mathfrak{l})$ as its Levi factor. Then $\mathcal{S}(\Xi)$ contains the following subspace.

Definition 3.4. $\Gamma(\Xi) := \{\zeta \in \mathcal{S}(\Xi) \mid (\mathrm{id} \otimes dL_x)\zeta = (S(x) \otimes \mathrm{id})\zeta, \ \forall x \in \mathrm{U}_q(\mathfrak{p})\}$.

Again by using the super-commutativity of the $\mathrm{U}_q(\mathfrak{g})$-actions dL and dR on $\mathcal{A}(\mathfrak{g})$ we can easily show that

Lemma 3.8. $\Gamma(\Xi)$ *is a* $\mathrm{U}_q(\mathfrak{g})$*-submodule of* $\mathcal{S}(\Xi)$*. Also a* $\mathrm{U}_q(\mathfrak{p})$*-module homomorphism* $\phi : \Xi \to \Xi'$ *induces a* $\mathrm{U}_q(\mathfrak{g})$*-equivariant map*

$$\Gamma(\phi) = \phi \otimes \mathrm{id} : \Gamma(\Xi) \to \Gamma(\Xi'). \tag{3.14}$$

Let $X(\mathfrak{p}, \mathfrak{l})$ denote the set of $E_{a+1,a}$ or $E_{a,a+1}$ which are contained in $\mathrm{U}_q(\mathfrak{p})$ but not in $\mathrm{U}_q(\mathfrak{l})$. If the $\mathrm{U}_q(\mathfrak{p})$-module Ξ has the property that every element of $X(\mathfrak{p}, \mathfrak{l})$ acts by zero, then in this case the definition of $\Gamma(\Xi)$ reduces to

$$\Gamma(\Xi) = \{\zeta \in \mathcal{S}(\Xi) \mid (\mathrm{id} \otimes dL_x)\zeta = 0, \ \forall x \in X(\mathfrak{p}, \mathfrak{l})\}.$$

Thus $\Gamma(\Xi)$ plays a similar role as the space of holomorphic sections in classical geometry. We shall refer to it as the space of holomorphic sections of the homogeneous supervector bundle determined by $\mathcal{S}(\Xi)$.

We shall promote Γ to a covariant functor from the category $\mathcal{C}(\mathfrak{p}, \mathfrak{l})$ of the $\mathrm{U}_q(\mathfrak{l})$-finite modules over the parabolic subalgebra $\mathrm{U}_q(\mathfrak{p})$ to the category $\mathcal{C}(\mathfrak{g}, \mathfrak{g})$ of locally finite $\mathrm{U}_q(\mathfrak{g})$-modules. The resultant functor is shown to be left exact, and its right derived functors will be regarded as the Dolbeault cohomology groups of the homogeneous supervector bundle.

4. Induction Functors

We study induction functors and their derived functors in this section. Results will be applied to develop a representation theoretical formulation of a quantum analogue of Dolbeault cohomology for the quantum homogeneous supervector bundles. References for background material on this section are [4] and [8]. The elementary facts from homological algebra used here can be found in any text book, e.g., [19].

4.1. **Categories of modules.** We start with a discussion on module categories of $\mathrm{U}_q(\mathfrak{g})$ and its quantum sub-superalgebras which will be used later.

For any quantum superalgebra U, we shall assume that every U-module to be considered in this paper is $\mathbb{Z}_2$-graded. Thus corresponding to each U-module V, there exists another module $\wp V$ which is equal to V as a set, but with $(\wp V)_{\bar{0}} = V_{\bar{1}}$, and $(\wp V)_{\bar{1}} = V_{\bar{0}}$. Let $\wp(v)$ denote the element of $\wp V$ corresponding to $v \in V$. The action of U on $\wp V$ is defined by $x\wp(v) = (-1)^{[x]}\wp(xv)$, for all $x \in \mathrm{U}$.

Let $\mathrm{U}_q(\mathfrak{r})$ be a quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$. Denote by $\mathcal{C}_{inh}(\mathfrak{r})$ the category of $\mathrm{U}_q(\mathfrak{r})$-modules, where the space $\mathrm{Hom}_{\mathrm{U}}(V, W)$ of morphisms between any two U-modules V and W is a $\mathbb{Z}_2$-graded subspace of $\mathrm{Hom}_{\mathbb{C}(q)}(V, W)$ consisting of such elements ϕ that for all $x \in \mathrm{U}_q(\mathfrak{r})$ and $v \in V$, $\phi(xv) = (-1)^{[x][\phi]}x\phi(v)$. The parity change map $\wp$ is odd, and becomes a covariant functor on $\mathcal{C}_{inh}(\mathfrak{r})$ if for any $\phi \in \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{r})}(V, W)$ we define $\wp(\phi) \in \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{r})}(\wp V, \wp W)$ to be the same as ϕ on sets. Note that if $\phi \in \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{r})}(V, W)$ is an inhomogeneous morphism between objects V and W in $\mathcal{C}_{inh}(\mathfrak{r})$, the kernel and image of ϕ are not necessarily $\mathbb{Z}_2$-graded in general, thus $\mathcal{C}_{inh}(\mathfrak{r})$ is not an Abelian category.

Assume $\mathrm{U}_q(\mathfrak{r})$ contains a reductive sub-superalgebra $\mathrm{U}_q(\mathfrak{k})$ of $\mathrm{U}_q(\mathfrak{g})$. Every $\mathrm{U}_q(\mathfrak{r})$-module V naturally restricts to a $\mathrm{U}_q(\mathfrak{k})$-module.

Definition 4.1. *The* $\mathrm{U}_q(\mathfrak{k})$*-finite subspace* $V[\mathrm{U}_q(\mathfrak{k})]$ *of* V *is defined to be the* $\mathbb{C}(q)$*-span of the integral weight vectors* $v \in V$ *satisfying* $\dim(\mathrm{U}_q(\mathfrak{k})v) < \infty$.

Here $\mathrm{U}_q(\mathfrak{k})v := \{xv \mid x \in \mathrm{U}_q(\mathfrak{k})\}$. Elements of $V[\mathrm{U}_q(\mathfrak{k})]$ will be called $\mathrm{U}_q(\mathfrak{k})$-finite. Also, a $\mathrm{U}_q(\mathfrak{r})$-module V is called $\mathrm{U}_q(\mathfrak{k})$-finite if $V = V[\mathrm{U}_q(\mathfrak{k})]$.

Remark 4.1. *If* V *is a* $\mathbb{Z}_2$*-graded* $\mathrm{U}_q(\mathfrak{k})$*-finite* $\mathrm{U}_q(\mathfrak{r})$*-module and* $\phi \in \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{r})}(V, W)$ *a* homogeneous *morphism, then* $\phi(V)$ *is a* $\mathbb{Z}_2$*-graded* $\mathrm{U}_q(\mathfrak{k})$*-finite* $\mathrm{U}_q(\mathfrak{r})$*-submodule of* W.

Let $\mathrm{U}_q(\mathfrak{q})$ be either a parabolic or reductive quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$. If $\mathrm{U}_q(\mathfrak{q})$ contains the reductive quantum sub-superalgebra $\mathrm{U}_q(\mathfrak{k})$, we shall talk about the pair $(\mathrm{U}_q(\mathfrak{q}), \mathrm{U}_q(\mathfrak{k}))$ of quantum sub-superalgebras. Two pairs of sub-superalgebras $(\mathrm{U}_q(\mathfrak{q}), \mathrm{U}_q(\mathfrak{k}))$ and $(\mathrm{U}_q(\mathfrak{p}), \mathrm{U}_q(\mathfrak{l}))$ are said to be compatible if we have the Hopf superalgebra inclusions $\mathrm{U}_q(\mathfrak{q}) \supseteq \mathrm{U}_q(\mathfrak{p})$ and $\mathrm{U}_q(\mathfrak{k}) \supseteq \mathrm{U}_q(\mathfrak{l})$, and in this case, we write $(\mathrm{U}_q(\mathfrak{q}), \mathrm{U}_q(\mathfrak{k})) \supseteq (\mathrm{U}_q(\mathfrak{p}), \mathrm{U}_q(\mathfrak{l}))$. Given a pair $(\mathrm{U}_q(\mathfrak{q}), \mathrm{U}_q(\mathfrak{k}))$, we shall denote by $\mathcal{C}_{inh}(\mathfrak{q}, \mathfrak{k})$ the full subcategory of $\mathcal{C}_{inh}(\mathfrak{q})$ with the $\mathrm{U}_q(\mathfrak{k})$-finite $\mathrm{U}_q(\mathfrak{q})$-modules as its objects. Clearly, $\mathcal{C}_{inh}(\mathfrak{q}, \mathfrak{k})$ is closed under passage to graded sub-modules, graded quotients and finite direct sums. It is also closed under finite tensor products.

Definition 4.2. *Let* $\mathcal{C}(\mathfrak{q})$ *be the subcategory of* $\mathcal{C}_{int}(\mathfrak{q})$ *consisting of the same objects and the* even *morphisms of* $\mathcal{C}_{inh}(\mathfrak{q})$*. Let* $\mathcal{C}(\mathfrak{q}, \mathfrak{k})$ *be the full subcategory of* $\mathcal{C}(\mathfrak{q})$ *with the* $\mathrm{U}_q(\mathfrak{k})$*-finite objects.*

Then $\mathcal{C}(\mathfrak{q})$ is obviously an Abelian category, and it follows from Remark 4.1 that $\mathcal{C}(\mathfrak{q}, \mathfrak{k})$ is also Abelian.

4.2. **Induction functors.** Let $(\mathrm{U}_q(\mathfrak{p}), \mathrm{U}_q(\mathfrak{l}))$ be a pair of quantum sub-superalgebras of $\mathrm{U}_q(\mathfrak{g})$, where $\mathrm{U}_q(\mathfrak{p})$ is either a parabolic or reductive quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$, and $\mathrm{U}_q(\mathfrak{l})$ is a reductive quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$ contained in $\mathrm{U}_q(\mathfrak{p})$. We recall that all the objects of the categories $\mathcal{C}(\mathfrak{p})$ and $\mathcal{C}(\mathfrak{p}, \mathfrak{l})$ are $\mathbb{Z}_2$-graded, and all the morphisms of the categories are even.

Definition 4.3. *Define a covariant functor* $Z_{\mathfrak{p}}^{\mathfrak{p},\mathfrak{l}} : \mathcal{C}(\mathfrak{p}) \to \mathcal{C}(\mathfrak{p}, \mathfrak{l})$ *in the following way: for any object* V, $Z_{\mathfrak{p}}^{\mathfrak{p},\mathfrak{l}}(V)$ *is the (not necessarily direct) sum of the* $\mathrm{U}_q(\mathfrak{l})$*-finite* $\mathbb{Z}_2$*-graded* $\mathrm{U}_q(\mathfrak{p})$*-submodules of* V*, and for any morphism* $\phi \in \mathrm{Hom}_{\mathcal{C}(\mathfrak{p})}(V, W)$, $Z_{\mathfrak{p}}^{\mathfrak{p},\mathfrak{l}}(\phi) = \phi|_{Z_{\mathfrak{p}}^{\mathfrak{p},\mathfrak{l}}(V)}$.

$Z_{\mathfrak{p}}^{\mathfrak{p},\mathfrak{l}}$ is well defined because of Remark 4.1. When $\mathrm{U}_q(\mathfrak{p}) = \mathrm{U}_q(\mathfrak{l})) = \mathrm{U}_q(\mathfrak{g})$, we have an analogue of the Zuckerman functor.

Lemma 4.1. *The functor* $Z_{\mathfrak{p}}^{\mathfrak{p},\mathfrak{l}} : \mathcal{C}(\mathfrak{p}) \to \mathcal{C}(\mathfrak{p}, \mathfrak{l})$ *is left exact.*

Proof. Even though the proof is straightforward, we nevertheless give the details here because of the importance of the functor $Z_{\mathfrak{p}}^{\mathfrak{p},\mathfrak{l}}$. Let us temporarily use Z to denote $Z_{\mathfrak{p}}^{\mathfrak{p},\mathfrak{l}}$. Given any exact sequence

$$0 \longrightarrow U \xrightarrow{i} V \xrightarrow{j} W$$

in $\mathcal{C}(\mathfrak{p})$, we want to show that the following sequence in $\mathcal{C}(\mathfrak{p}, \mathfrak{l})$ is also exact:

$$0 \longrightarrow Z(U) \xrightarrow{Z(i)} Z(V) \xrightarrow{Z(j)} Z(W).$$

Assume U' is a $\mathrm{U}_q(\mathfrak{l})$-finite $\mathrm{U}_q(\mathfrak{p})$-submodule of U. Then $i(U')$ is a $\mathrm{U}_q(\mathfrak{l})$-finite $\mathrm{U}_q(\mathfrak{p})$-submodule of V. Thus the injectivity of i implies the injectivity of $Z(i)$.

Let V' be a $\mathrm{U}_q(\mathfrak{p})$-submodule of $Z(V)$. If an element $v \in V'$ is in $KerZ(j)$, then there exists a unique $u \in U$ such that $v = i(u)$. Now $\mathrm{U}_q(\mathfrak{p})v = i\left(\mathrm{U}_q(\mathfrak{p})u\right)$ is a $\mathrm{U}_q(\mathfrak{l})$-finite $\mathrm{U}_q(\mathfrak{p})$-submodule of V. The injectivity of i forces $\mathrm{U}_q(\mathfrak{p})u$ to be a $\mathrm{U}_q(\mathfrak{p})$-submodule of $Z(U)$. In particular, $u \in Z(U)$. Thus $ImZ(i) \supseteq KerZ(j)$. But it is obvious that $ImZ(i) \subseteq KerZ(j)$. Hence the sequence is also exact at $Z(V)$. □

Let $\mathrm{U}_q(\mathfrak{q})$ either be a parabolic or reductive quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$, and let $(\mathrm{U}_q(\mathfrak{p}), \mathrm{U}_q(\mathfrak{l}))$ be as given above with $\mathrm{U}_q(\mathfrak{q}) \supseteq \mathrm{U}_q(\mathfrak{p})$. We define the covariant functor $I^{\mathfrak{q}}_{\mathfrak{p},\mathfrak{l}} : \mathcal{C}(\mathfrak{p}, \mathfrak{l}) \to \mathcal{C}(\mathfrak{q})$ by

$$\mathrm{I}^{\mathfrak{q}}_{\mathfrak{p},\mathfrak{l}}(V) := \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{q}),\, V), \qquad \mathrm{I}^{\mathfrak{q}}_{\mathfrak{p},\mathfrak{l}}(\phi) := \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{q}),\, \phi) \tag{4.1}$$

for any object V and morphism $\phi \in \mathrm{Hom}_{\mathcal{C}(\mathfrak{p},\mathfrak{l})}(V,\, W)$. Here $\mathrm{I}^{\mathfrak{q}}_{\mathfrak{p},\mathfrak{l}}(\phi)$ is defined for any $\zeta \in \mathrm{I}^{\mathfrak{q}}_{\mathfrak{p},\mathfrak{l}}(V)$ by

$$\langle \mathrm{I}^{\mathfrak{q}}_{\mathfrak{p},\mathfrak{l}}(\phi)(\zeta),\, x\rangle \;=\; \phi(\langle \zeta,\, x\rangle), \quad \forall x \in \mathrm{U}_q(\mathfrak{q}).$$

Note that $\langle \zeta,\, x\rangle \in V$. The $\mathrm{U}_q(\mathfrak{q})$ action on $\mathrm{I}^{\mathfrak{q}}_{\mathfrak{p},\mathfrak{l}}(V)$

$$\mathrm{U}_q(\mathfrak{q}) \otimes \mathrm{I}^{\mathfrak{q}}_{\mathfrak{p},\mathfrak{l}}(V) \to \mathrm{I}^{\mathfrak{q}}_{\mathfrak{p},\mathfrak{l}}(V), \qquad y \otimes \zeta \mapsto y \circ \zeta, \tag{4.2}$$

is defined by $\langle y \circ \zeta,\, x\rangle = (-1)^{[y]([x]+[\zeta])}\langle \zeta,\, xy\rangle$, for all $x \in \mathrm{U}_q(\mathfrak{q})$. The functor $I^{\mathfrak{q}}_{\mathfrak{p},\mathfrak{l}}$ is the composition of the exact functor of tensoring with $\mathrm{U}_q(\mathfrak{q})^*$ and the left exact functor of taking $\mathrm{U}_q(\mathfrak{p})$ invariant submodules, thus is also left exact.

Definition 4.4. *Given compatible pairs* $(\mathrm{U}_q(\mathfrak{q}), \mathrm{U}_q(\mathfrak{k})) \supseteq (\mathrm{U}_q(\mathfrak{p}), \mathrm{U}_q(\mathfrak{l}))$*, we introduce the covariant functor*

$$\mathrm{I}^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{p},\mathfrak{l}} := Z^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{q}} \circ \mathrm{I}^{\mathfrak{q}}_{\mathfrak{p},\mathfrak{l}} :\, \mathcal{C}(\mathfrak{p}, \mathfrak{l}) \to \mathcal{C}(\mathfrak{q}, \mathfrak{k}),$$

and call it the induction functor from $\mathcal{C}(\mathfrak{p}, \mathfrak{l})$ *to* $\mathcal{C}(\mathfrak{q}, \mathfrak{k})$.

Lemma 4.2. *The induction functor* $\mathrm{I}^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{p},\mathfrak{l}}$ *is left exact.*

Proof. Since both $\mathrm{I}^{\mathfrak{q}}_{\mathfrak{p},\mathfrak{l}}$ and $Z^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{q}}$ are left exact, their composition must also be left exact. □

Let us examine some further properties of the Zuckerman functor and the induction functor.

Let $\mathrm{U}_q(\mathfrak{q})$ be either a parabolic or reductive quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$, and let $\mathrm{U}_q(\mathfrak{k})$ be a reductive quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$. Assume $\mathrm{U}_q(\mathfrak{q}) \supset \mathrm{U}_q(\mathfrak{k})$. Then for any object W of $\mathcal{C}(\mathfrak{q})$,

$$Z^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{q}}(W) \;=\; W[\mathrm{U}_q(\mathfrak{k})]. \tag{4.3}$$

To prove this, we define the adjoint action of $\mathrm{U}_q(\mathfrak{q})$ on itself

$$\mathrm{ad}_y(x) \;=\; \sum_{(y)} (-1)^{[y_{(2)}][x]} y_{(1)} x S(y_{(2)}), \quad x, y \in \mathrm{U}_q(\mathfrak{q}),$$

where Sweedler's notation $\Delta(y) = \sum_{(y)} y_{(1)} \otimes y_{(2)}$ is used for the co-multiplication of y. By using the Poincaré-Birkhoff-Witt (PBW) Theorem 2.1, we can choose a set of y_i each of which is a product of E_{ab} associated with the roots of $\mathfrak{q}$ not contained in $\mathfrak{k}$, such that every element $x \in \mathrm{U}_q(\mathfrak{q})$ can be expressed as a finite sum $x = \sum y_i u_i$ with $u_i \in \mathrm{U}_q(\mathfrak{k})$. By considering the PBW theorem again, we see that there exists a finite set Y_x of the y_i such that every element of the space $\mathrm{ad}_{\mathrm{U}_q(\mathfrak{k})}(x) := \{\mathrm{ad}_u(x) \mid u \in \mathrm{U}_q(\mathfrak{k})\}$

can be expressed in the form $\sum y_i u_i'$ with $y_i \in Y_x$ and $u_i' \in \mathrm{U}_q(\mathfrak{k})$. If $w \in W[\mathrm{U}_q(\mathfrak{k})]$, then

$$u(xw) \;=\; \sum_{(u)} (-1)^{[u_{(2)}][x]} \left(\mathrm{ad}_{u_{(1)}}(x)\right) \left(u_{(2)} w\right), \quad x \in \mathrm{U}_q(\mathfrak{q}), u \in \mathrm{U}_q(\mathfrak{k}).$$

This implies $u(xw) \in \sum_{y \in Y_x} y\left(\mathrm{U}_q(\mathfrak{k})(xw)\right)$, for all $u \in \mathrm{U}_q(\mathfrak{k})$. Therefore,

$$\dim\left(\mathrm{U}_q(\mathfrak{k})(xw)\right) \le |Y_x| \dim\left(\mathrm{U}_q(\mathfrak{k}) w\right) < \infty,$$

where Y_x is the cardinality of Y_x. Also, if x carries a fixed weight and w is a weight vector of $W[\mathrm{U}_q(\mathfrak{k})]$, then xw is a weight vector of $W[\mathrm{U}_q(\mathfrak{k})]$ with integral weight. Hence $W[\mathrm{U}_q(\mathfrak{k})]$ is indeed a $\mathrm{U}_q(\mathfrak{q})$-submodule of W.

Lemma 4.3. *Given compatible pairs* $(\mathrm{U}_q(\mathfrak{r}), \mathrm{U}_q(\mathfrak{j}))) \supseteq (\mathrm{U}_q(\mathfrak{q}), \mathrm{U}_q(\mathfrak{k})) \supseteq (\mathrm{U}_q(\mathfrak{p}), \mathrm{U}_q(\mathfrak{l}))$ *of quantum sub-superalgebras of* $\mathrm{U}_q(\mathfrak{g})$*, we have* $\mathrm{I}^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{q},\mathfrak{k}} \circ \mathrm{I}^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{p},\mathfrak{l}} = \mathrm{I}^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{p},\mathfrak{l}}$ *as covariant functors from* $\mathcal{C}(\mathfrak{p},\mathfrak{l})$ *to* $\mathcal{C}(\mathfrak{r},\mathfrak{j})$.

Proof. It is clearly true that for any morphism $\phi \in \mathrm{Hom}_{\mathcal{C}(\mathfrak{p},\mathfrak{l})}(V, V')$, we have $\mathrm{I}^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{q},\mathfrak{k}} \circ \mathrm{I}^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{p},\mathfrak{l}}(\phi)$ $= \mathrm{I}^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{p},\mathfrak{l}}(\phi)$. To prove that the Lemma holds on objects, we need the following technical result which will be proved below: if $(\mathrm{U}_q(\mathfrak{r}), \mathrm{U}_q(\mathfrak{j})) \supseteq (\mathrm{U}_q(\mathfrak{q}), \mathrm{U}_q(\mathfrak{k}))$, then for any object W of $\mathcal{C}(\mathfrak{q})$,

$$Z^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{r}} \circ \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}\left(\mathrm{U}_q(\mathfrak{r}), W\right) \;=\; Z^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{r}} \circ \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}\left(\mathrm{U}_q(\mathfrak{r}),\, Z^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{q}}(W)\right). \tag{4.4}$$

With (4.4) granted, we have for any object V of $\mathcal{C}(\mathfrak{p},\mathfrak{l})$ that

$$\begin{aligned} \mathrm{I}^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{q},\mathfrak{k}} \circ \mathrm{I}^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{p},\mathfrak{l}}(V) &= Z^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{r}} \circ \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}\left(\mathrm{U}_q(\mathfrak{r}),\, \mathrm{I}^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{p},\mathfrak{l}}(V)\right) \\ &= Z^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{r}} \circ \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}\left(\mathrm{U}_q(\mathfrak{r}),\, \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{q}), V)\right). \end{aligned}$$

The far right hand side of the equation can be simplified by using the following relation

$$\begin{aligned} \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}\left(\mathrm{U}_q(\mathfrak{r}), \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{q}),\, V)\right) &= \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}\left(\mathrm{U}_q(\mathfrak{q}) \otimes_{\mathrm{U}_q(\mathfrak{q})} \mathrm{U}_q(\mathfrak{r}),\, V\right) \\ &= \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}\left(\mathrm{U}_q(\mathfrak{r}),\, V\right), \end{aligned}$$

and we arrive at

$$\mathrm{I}^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{q},\mathfrak{k}} \circ \mathrm{I}^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{p},\mathfrak{l}}(V) = Z^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{r}} \circ \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}\left(\mathrm{U}_q(\mathfrak{r}),\, V\right) = \mathrm{I}^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{p},\mathfrak{l}}(V).$$

Now we turn to the proof of equation (4.4), which is equivalent to the statement that for any $\zeta \in Z^{\mathfrak{r},\mathfrak{j}}_{\mathfrak{r}} \circ \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}\left(\mathrm{U}_q(\mathfrak{r}), W\right)$,

$$\langle \zeta,\ z\rangle \in Z^{\mathfrak{q},\mathfrak{k}}_{\mathfrak{q}}(W), \qquad \forall z \in \mathrm{U}_q(\mathfrak{r}). \tag{4.5}$$

By the PBW theorem for $\mathrm{U}_q(\mathfrak{r})$, there exists a set of x_i, each of which is a product of elements associated with roots of $\mathfrak{r}$ not contained in $\mathfrak{q}$, such that every $z \in \mathrm{U}_q(\mathfrak{r})$ can be expressed as a finite sum $\sum y_i x_i$ with $y_i \in \mathrm{U}_q(\mathfrak{q})$. Let ν_i be the weight of x_i, which is a sum of roots of $\mathfrak{r}$ thus is integral. We have

$$\langle u \circ \zeta,\, x_i\rangle \;=\; \sum_{(u)} (-1)^{[\zeta][u_{(2)}]} \pi_W(u_{(1)}) \langle \zeta,\, \mathrm{ad}_{S(u_{(2)})}(x_i)\rangle, \quad u \in \mathrm{U}_q(\mathfrak{k}),$$

where $u \circ \zeta$ is as defined by (4.2), and π_W refers to the $\mathrm{U}_q(\mathfrak{q})$ action on W. From this equation we can deduce that

$$\pi_W(u) \langle \zeta,\, x_i\rangle \;=\; \sum_{(u)} (-1)^{[u_{(2)}][\zeta]} \langle u_{(1)} \circ \zeta,\, \mathrm{ad}_{u_{(2)}}(x_i)\rangle, \quad u \in \mathrm{U}_q(\mathfrak{k}).$$

Arguing as in the proof of (4.3), we conclude that for every x_i, there exists a finite set X_i of the x_j such that every element of $\mathrm{ad}_{\mathrm{U}_q(\mathfrak{k})}(x_i)$ can be expressed as $\sum x_j u_j$, where $x_j \in X_i$ and $u_j \in \mathrm{U}_q(\mathfrak{k})$. Now $\langle \zeta, \sum x_j u_j\rangle = \sum \langle u_j \circ \zeta, \sum x_j\rangle$, and $\zeta \in Z_{\mathfrak{r}}^{\mathfrak{r},\mathfrak{j}} \circ \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}(\mathrm{U}_q(\mathfrak{r}), W)$ implies that the span of $u \circ \zeta$ for all $u \in \mathrm{U}_q(\mathfrak{k})$ is finite dimensional. Therefore,

$$\dim\left(\pi_W(\mathrm{U}_q(\mathfrak{k}))\langle \zeta, x_i\rangle\right) < \infty, \text{ that is, } \langle \zeta, x_i\rangle \in Z_{\mathfrak{q}}^{\mathfrak{q},\mathfrak{k}}(W), \forall i.$$

Since $Z_{\mathfrak{q}}^{\mathfrak{q},\mathfrak{k}}(W)$ is $\mathrm{U}_q(\mathfrak{q})$-stable, we have $\sum_i \pi_W(\mathrm{U}_q(\mathfrak{q}))\langle \zeta, x_i\rangle \subset Z_{\mathfrak{q}}^{\mathfrak{q},\mathfrak{k}}(W)$. Now every element of $\mathrm{U}_q(\mathfrak{r})$ can be expressed as $\sum y_i x_i$ with $y_i \in \mathrm{U}_q(\mathfrak{q})$. By the definition of $Z_{\mathfrak{r}}^{\mathfrak{r},\mathfrak{j}} \circ \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}(\mathrm{U}_q(\mathfrak{r}), W)$, we have $\sum\langle \zeta, y_i x_i\rangle = \sum(-1)^{[y_i][\zeta]}\pi_W(y_i)\langle \zeta, x_i\rangle$, where the right hand side has just been shown to belong to $Z_{\mathfrak{q}}^{\mathfrak{q},\mathfrak{k}}(W)$. This proves equation (4.5), thus completes the proof of the Lemma. □

Denote by $\mathcal{F}_{\mathfrak{q},\mathfrak{k}}^{\mathfrak{p},\mathfrak{l}} : \mathcal{C}(\mathfrak{q},\mathfrak{k}) \to \mathcal{C}(\mathfrak{p},\mathfrak{l})$ the forgetful functor. We shall refer to the next theorem as Frobenius reciprocity, which in particular implies that the induction functor $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}$ is right adjoint to the forgetful functor $\mathcal{F}_{\mathfrak{q},\mathfrak{k}}^{\mathfrak{p},\mathfrak{l}}$. The theorem plays a crucial role in the study of derived functors of the induction functors.

Theorem 4.1. *There exists the natural even isomorphism*

$$\begin{array}{rcl}\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}\left(W, \mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}(V)\right) & \stackrel{\sim}{\longrightarrow} & \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}\left(\mathcal{F}_{\mathfrak{q},\mathfrak{k}}^{\mathfrak{p},\mathfrak{l}}(W), V\right), \\ \phi & \mapsto & \phi(\mathbb{1}_{\mathrm{U}_q(\mathfrak{q})}),\end{array} \tag{4.6}$$

of $\mathbb{Z}_2$-graded vector spaces for all W in $\mathcal{C}(\mathfrak{q},\mathfrak{k})$ and V in $\mathcal{C}(\mathfrak{p},\mathfrak{l})$.

Proof. Since W is an object of $\mathcal{C}(\mathfrak{q},\mathfrak{k})$, the image of any vector of W under an arbitrary $\phi \in \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}\left(W, \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{q}), V)\right)$ belongs to $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}(V)$. From this we can easily deduce that

$$\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}(W, \mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}(V)) \cong \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}\left(W, \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{q}), V)\right).$$

The right hand side can be further rewritten as $\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}\left(\mathrm{U}_q(\mathfrak{q}) \otimes_{\mathrm{U}_q(\mathfrak{q})} W, V\right)$. Now $\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}\left(\mathrm{U}_q(\mathfrak{q}) \otimes_{\mathrm{U}_q(\mathfrak{q})} W, V\right) \cong \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}\left(\mathcal{F}_{\mathfrak{q},\mathfrak{k}}^{\mathfrak{p},\mathfrak{l}}(W), V\right)$. Thus

$$\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}\left(W, \mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}(V)\right) \cong \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}\left(\mathcal{F}_{\mathfrak{q},\mathfrak{k}}^{\mathfrak{p},\mathfrak{l}}(W), V\right). \tag{4.7}$$

This establishes the claimed isomorphism between the vector spaces. Let us now show that $\phi \mapsto \phi(\mathbb{1}_{\mathrm{U}_q(\mathfrak{q})})$, $\phi \in \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}(W, \mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}(V))$, is indeed the required map. The $\phi(\mathbb{1}_{\mathrm{U}_q(\mathfrak{q})})$ (We shall write 1 for the identity $\mathbb{1}_{\mathrm{U}_q(\mathfrak{q})}$ of $\mathrm{U}_q(\mathfrak{q})$.) is the evaluation of ϕ at the identity of $\mathrm{U}_q(\mathfrak{q})$. Denote by $\circ$ the action of $\mathrm{U}_q(\mathfrak{q})$ on $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}(V)$. Then for any $x \in \mathrm{U}_q(\mathfrak{q})$ and $w \in W$, we have

$$\phi(1)(xw) = (-1)^{[x][\phi]}(x \circ \phi)(1)(w) = \phi(x)(w),$$

where the symbol $\circ$ refers to the $\mathrm{U}_q(\mathfrak{q})$-action on $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}(V)$. If ϕ belongs to the kernel of the map (4.6), then $\phi(x) = 0$, $\forall x \in \mathrm{U}_q(\mathfrak{q})$. This forces $\phi = 0$. Thus the map (4.6) is injective, and because of the isomorphism (4.7), it must be a bijection.

We still need to show that $\phi(1) \in \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathcal{F}_{\mathfrak{q},\mathfrak{k}}^{\mathfrak{p},\mathfrak{l}}(W), V)$. But this is clear, as the defining property of $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}(V)$ implies

$$\phi(x)(w) = (-1)^{[x][\phi]}x\left(\phi(1)(w)\right), \quad \forall x \in \mathrm{U}_q(\mathfrak{p}).$$

This completes the proof. □

The following result is an immediate consequence of Theorem 4.1.

Corollary 4.1. *The induction functor* $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}$ *takes injectives to injectives.*

Proof. If V is an injective object in $\mathcal{C}(\mathfrak{p},\mathfrak{l})$, then $\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\,\cdot\,, V)$ is an exact functor from $\mathcal{C}(\mathfrak{p},\mathfrak{l})$ to the category of $\mathbb{Z}_2$-graded vector spaces. Thus by the Frobenius reciprocity of Theorem 4.1, $\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{q})}\left(\,\cdot\,, \mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}(V)\right)$ is exact on $\mathcal{C}(\mathfrak{q},\mathfrak{k})$. This in turn implies that $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}(V)$ is injective in $\mathcal{C}(\mathfrak{q},\mathfrak{k})$. □

Consider the pair $(\mathrm{U}_q(\mathfrak{p}), \mathrm{U}_q(\mathfrak{l}))$ of quantum sub-superalgebras of $\mathrm{U}_q(\mathfrak{g})$, where $\mathrm{U}_q(\mathfrak{l})$ is assumed to be reductive as always.

Corollary 4.2. *The category* $\mathcal{C}(\mathfrak{p},\mathfrak{l})$ *has enough injectives.*

Proof. Let $\mathrm{U_q}(\mathfrak{l}_0) = \mathrm{U_q}(\mathfrak{l}) \cap \mathrm{U_q}(\mathfrak{g}_0)$. Every $\mathrm{U_q}(\mathfrak{l}_0)$-finite $\mathrm{U}_q(\mathfrak{p})$-module is also $\mathrm{U}_q(\mathfrak{l})$-finite and vice versa, hence the categories $\mathcal{C}(\mathfrak{p},\mathfrak{l}_0)$ and $\mathcal{C}(\mathfrak{p},\mathfrak{l})$ are identical. Since $\mathrm{U_q}(\mathfrak{l}_0)$ is the tensor product of some non-super $\mathrm{U_q}(\mathfrak{gl}_\mathrm{k})$'s, every object of $\mathcal{C}(\mathfrak{l}_0,\mathfrak{l}_0)$ is semi-simple, thus is injective. Let V be an $\mathrm{U}_q(\mathfrak{l})$-finite $\mathrm{U}_q(\mathfrak{p})$-module, which can be restricted to an object of $\mathcal{C}(\mathfrak{l}_0,\mathfrak{l}_0)$. Now $\mathrm{I}_{\mathfrak{l}_0,\mathfrak{l}_0}^{\mathfrak{p},\mathfrak{l}}(V)$ is injective as follows from the above Corollary. By Theorem 4.1 we have the isomorphism

$$F:\ \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}\left(V,\, \mathrm{I}_{\mathfrak{l}_0,\mathfrak{l}_0}^{\mathfrak{p},\mathfrak{l}}(V)\right) \ \xrightarrow{\ \sim\ }\ \mathrm{Hom}_{\mathrm{U_q}(\mathfrak{l}_0)}(V,\, V)\,.$$

Consider the pre-image of the identity map $\mathrm{id}_V \in \mathrm{Hom}_{\mathrm{U_q}(\mathfrak{l}_0)}(V,\, V)$ under F,

$$\iota := F^{-1}(\mathrm{id}_V):\ V \to \mathrm{I}_{\mathfrak{l}_0,\mathfrak{l}_0}^{\mathfrak{p},\mathfrak{l}}(V), \qquad v \mapsto \iota_v, \tag{4.8}$$

which is an injective $\mathrm{U}_q(\mathfrak{p})$-map. It satisfies $\iota_v(x) = (-1)^{[x][v]} x v,\ \ \forall x \in \mathrm{U}_q(\mathfrak{p})$. □

Remark 4.2. *The* $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{q},\mathfrak{k}}$ *can be extended to a covariant functor* $\mathcal{C}_{inh}(\mathfrak{q},\mathfrak{k}) \to \mathcal{C}_{inh}(\mathfrak{p},\mathfrak{l})$ *in the obvious way. It also takes injectives to injectives. By using Frobenius reciprocity, we can also show that the category* $\mathcal{C}_{inh}(\mathfrak{p},\mathfrak{l})$ *has enough injectives.*

Now we restrict our attention to the induction functor $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}: \mathcal{C}(\mathfrak{p},\mathfrak{l}) \to \mathcal{C}(\mathfrak{g},\mathfrak{g})$. Since the Abelian category $\mathcal{C}(\mathfrak{p},\mathfrak{l})$ has enough injectives, and $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}$ is left exact, it makes sense to talk about its right derived functors [19] $\left(\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}\right)^k$, $k \in \mathbb{Z}_+$. We now give a concrete description of $\left(\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}\right)^k$. Let V be any object of $\mathcal{C}(\mathfrak{p},\mathfrak{l})$. Then its restriction to a $\mathrm{U}_q(\mathfrak{l}_0)$-module lies in $\mathcal{C}(\mathfrak{l}_0,\mathfrak{l}_0)$ and thus is injective. We construct the following injective resolution of V in $\mathcal{C}(\mathfrak{p},\mathfrak{l})$,

$$0 \to V \xrightarrow{\ \iota\ } I^0(V) \xrightarrow{\ \delta^0\ } I^1(V) \xrightarrow{\ \delta^1\ } I^2(V) \xrightarrow{\ \delta^2\ } \cdots \tag{4.9}$$

where the $\mathrm{U}_q(\mathfrak{p})$-modules and maps are defined inductively by

$$\begin{aligned} I^{k+1}(V) &:= \mathrm{I}_{\mathfrak{l}_0,\mathfrak{l}_0}^{\mathfrak{p},\mathfrak{l}}(I^k(V)/\delta^{k-1}(I^{k-1}(V))), \\ \delta^k &:= \iota \circ p : I^k(V) \xrightarrow{p} I^k(V)/\delta^{k-1}(I^{k-1}(V)) \xrightarrow{\iota} I^{k+1}(V). \end{aligned} \tag{4.10}$$

Here ι is similarly defined as in (4.8), p is the canonical projection, and

$$I^0(V) = \mathrm{I}_{\mathfrak{l}_0,\mathfrak{l}_0}^{\mathfrak{p},\mathfrak{l}}(V), \qquad \delta^{-1} = \iota.$$

The sequence (4.9) is clearly a resolution, with all $I^k(V)$ being injective because of Corollary 4.1. We shall call this injective resolution the *standard resolution*. Now

apply the left exact covariant functor $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}$ to it and ignore the first term $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}(V)$, we arrive at the following complex in $\mathcal{C}(\mathfrak{g},\mathfrak{g})$:

$$0\to\Omega^0(\mathfrak{p},\mathfrak{l};V) \xrightarrow{d^0} \Omega^1(\mathfrak{p},\mathfrak{l};V) \xrightarrow{d^1} \Omega^2(\mathfrak{p},\mathfrak{l};V) \xrightarrow{d^2} \cdots , \tag{4.11}$$

where

$$\Omega^k(\mathfrak{p},\mathfrak{l};V) := \mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}(I^k(V)), \quad d^k := \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}(\mathrm{U}_q(\mathfrak{g}),\, \delta^k)\big|_{\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}(I^k(V))}.$$

Let us denote by $\Omega(\mathfrak{p},\mathfrak{l};V)$ the complex (4.11), and by $H^k(\Omega(\mathfrak{p},\mathfrak{l};V))$ its cohomology groups. Then we have [19]

$$\left(\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}\right)^k(V) = H^k(\Omega(\mathfrak{p},\mathfrak{l};V)), \quad k=0,1,...,$$

which are independent of the injective resolution (4.9) chosen. Left exactness of the induction functor implies

$$H^0(\Omega(\mathfrak{p},\mathfrak{l};V)) \ = \ \mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}(V).$$

5. Quantum Bott-Borel-Weil Theorem

Throughout this section, we shall assume that $(\mathrm{U}_q(\mathfrak{p}),\mathrm{U}_q(\mathfrak{l}))$ is a pair of quantum sub-superalgebras such that $\mathrm{U}_q(\mathfrak{l})$ is a reductive quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$, and $\mathrm{U}_q(\mathfrak{p})$ is the parabolic with $\mathrm{U}_q(\mathfrak{l})$ being its Levi factor. For the sake of concreteness, we also assume that $\mathrm{U}_q(\mathfrak{p})$ contains the lower triangular Borel subalgebra $\mathrm{U_q}(\bar{\mathfrak{b}})$ of $\mathrm{U}_q(\mathfrak{g})$. We shall also denote $\mathrm{U_q}(\mathfrak{p}_0)=\mathrm{U_q}(\mathfrak{g}_0)\cap\mathrm{U_q}(\mathfrak{p}_0)$.

The main results of the section are Theorems 5.2 and 5.3, which might be considered as a form of Bott-Borel-Weil theorem for the quantum general linear supergroup.

5.1. **Dolbeault cohomology groups.** We first formulate the Dolbeault cohomology groups of the homogeneous supervector bundles as the derived functor of an induction functor. First note that the domain of $\mathcal{S}$ can be extended to the category $\mathcal{C}(\mathfrak{l},\mathfrak{l})$, and that of Γ to the category $\mathcal{C}(\mathfrak{p},\mathfrak{l})$. Below we shall consider these more generally defined $\mathcal{S}$ and Γ. We have the following result.

Theorem 5.1. *$\mathcal{S}$ coincides with $\mathrm{I}_{\mathfrak{l},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}$ on $\mathcal{C}(\mathfrak{l},\mathfrak{l})$.*

Proof. By examining the second part of Proposition 3.1, we easily see that $\mathcal{S}$ agrees with $\mathrm{I}_{\mathfrak{l},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}$ on maps. Let Ξ be an object of $\mathcal{C}(\mathfrak{l},\mathfrak{l})$. The inclusion $\mathcal{S}(\Xi)\subseteq\mathrm{I}_{\mathfrak{l},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}(\Xi)$ is obvious since $\Xi\otimes\mathcal{A}(\mathfrak{g})$ is $\mathrm{U}_q(\mathfrak{g})$-finite with respect to the action $\mathrm{id}_\Xi\otimes dR_{\mathrm{U}_q(\mathfrak{g})}$. Any element $\zeta\in\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{l})}(\mathrm{U}_q(\mathfrak{g}),\,\Xi)$ can be expressed as $\zeta=\sum\xi_i\otimes f_i$, where $f_i\in\mathrm{U}_q(\mathfrak{g})^*$ and $\zeta_i\in\Xi$. The ζ belongs to $\mathrm{I}_{\mathfrak{l},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}(\Xi)$ only if $\dim(R_{\mathrm{U}_q(\mathfrak{g})}(f_i))<\infty$, $\forall i$. This is equivalent to the condition that all the f_i belong to $\mathrm{U}_q(\mathfrak{g})^\circ$, as follows from Lemma 3.1. Since Ξ regarded as a $\mathrm{U}_q(\mathfrak{h})$-module is integral, we may assume that the ξ_i are weight vectors with integral weights. The defining property of $\mathrm{I}_{\mathfrak{l},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}(\Xi)$ requires the f_i be $dL_{\mathrm{U}_q(\mathfrak{h})}$ eigenvectors in $\mathrm{U}_q(\mathfrak{g})^\circ$ with integral weights. Hence by Remark 3.1, the f_i must all belong to $\mathcal{A}(\mathfrak{g})$. □

In exactly the same manner we can show that

Corollary 5.1. $\Gamma(\Xi)=\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}(\Xi)$ *on $\mathcal{C}(\mathfrak{p},\mathfrak{l})$.*

Remark 5.1. *In view of the Theorem and this Corollary, we regard the right derived functors of $\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}$ as a form of Dolbeault cohomology of the quantum homogeneous super vector bundles. Thus we shall use the more suggestive notation $H^{0,k}(G/P,\,\mathcal{S}(\Xi))$ to denote $H^k(\Omega(\mathfrak{p},\mathfrak{l};\Xi))$.*

5.2. **The computation of cohomology groups.** The rest of the paper is devoted to the computation of $H^{0,k}(G/P, \mathcal{S}(\Xi))$.

5.2.1. *A special case with* $\mathrm{U}_q(\mathfrak{l}) \subset \mathrm{U}_\mathrm{q}(\mathfrak{g}_0)$. Denote by

$$\begin{aligned}
\mathcal{F}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_{\leq 0},\mathfrak{g}_0} : \mathcal{C}(\mathfrak{g}_{\leq 0}, \mathfrak{g}_0) &\rightarrow \mathcal{C}(\mathfrak{g}_0, \mathfrak{g}_0),\\
\mathcal{F}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{g}_{\leq 0},\mathfrak{l}} : \mathcal{C}(\mathfrak{g}_{\leq 0}, \mathfrak{l}) &\rightarrow \mathcal{C}(\mathfrak{g}_0, \mathfrak{l}),\\
\mathcal{F}^{\mathfrak{p}_0,\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}} : \mathcal{C}(\mathfrak{p}, \mathfrak{l}) &\rightarrow \mathcal{C}(\mathfrak{p}_0, \mathfrak{l}),
\end{aligned}$$

the forgetful functors.

Lemma 5.1. *If* $\mathrm{U}_q(\mathfrak{l}) \subset \mathrm{U}_q(\mathfrak{g}_0)$*, then we have the following relations:*

$$\mathcal{F}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_{\leq 0},\mathfrak{g}_0} \circ \mathrm{I}^{\mathfrak{g}_{\leq 0},\mathfrak{g}_0}_{\mathfrak{g}_{\leq 0},\mathfrak{l}} = \mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_0,\mathfrak{l}} \circ \mathcal{F}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{g}_{\leq 0},\mathfrak{l}}, \tag{5.1}$$

$$\mathcal{F}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{g}_{\leq 0},\mathfrak{l}} \circ \mathrm{I}^{\mathfrak{g}_{\leq 0},\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}} = \mathrm{I}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{p}_0,\mathfrak{l}} \circ \mathcal{F}^{\mathfrak{p}_0,\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}. \tag{5.2}$$

Proof. The first relation can be confirmed by directly checking the functors involved on objects and morphisms. For any object V of $\mathcal{C}(\mathfrak{g}_{\leq 0}, \mathfrak{l})$, $\mathrm{I}^{\mathfrak{g}_{\leq 0},\mathfrak{g}_0}_{\mathfrak{g}_{\leq 0},\mathfrak{l}}(V) = V[\mathrm{U}_\mathrm{q}(\mathfrak{g}_0)]$, and thus $\mathcal{F}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_{\leq 0},\mathfrak{g}_0} \circ \mathrm{I}^{\mathfrak{g}_{\leq 0},\mathfrak{g}_0}_{\mathfrak{g}_{\leq 0},\mathfrak{l}}(V)$ is $V[\mathrm{U}_\mathrm{q}(\mathfrak{g}_0)]$ regarded as a $\mathrm{U}_\mathrm{q}(\mathfrak{g}_0)$-module. Also, $\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_0,\mathfrak{l}} \circ \mathcal{F}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{g}_{\leq 0},\mathfrak{l}}(V) = \mathcal{F}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{g}_{\leq 0},\mathfrak{l}}(V)[\mathrm{U}_\mathrm{q}(\mathfrak{g}_0)]$, which is again $V[\mathrm{U}_\mathrm{q}(\mathfrak{g}_0)]$ regarded as a $\mathrm{U}_\mathrm{q}(\mathfrak{g}_0)$-module. Now for any $\phi \in \mathrm{Hom}_{\mathcal{C}(\mathfrak{g}_{\leq 0},\mathfrak{l})}(V, W)$, and all $v \in V[\mathrm{U}_\mathrm{q}(\mathfrak{g}_0)]$, we have $\mathcal{F}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_{\leq 0},\mathfrak{g}_0} \circ \mathrm{I}^{\mathfrak{g}_{\leq 0},\mathfrak{g}_0}_{\mathfrak{g}_{\leq 0},\mathfrak{l}}(\phi)(v) = \phi(v) = \mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_0,\mathfrak{l}} \circ \mathcal{F}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{g}_{\leq 0},\mathfrak{l}}(v)$, which belongs to $W[\mathrm{U}_\mathrm{q}(\mathfrak{g}_0)]$.

Now consider the second relation, which obviously holds on morphisms. By using the quantum PBW theorem, we can easily show that $\mathrm{U}_\mathrm{q}(\mathfrak{g}_0)/\mathrm{U}_\mathrm{q}(\mathfrak{p}_0) \cong \mathrm{U}_\mathrm{q}(\mathfrak{g}_{\leq 0})/\mathrm{U}_\mathrm{q}(\mathfrak{p})$ under the given conditions on $\mathrm{U}_q(\mathfrak{l})$ and $\mathrm{U}_q(\mathfrak{p})$. Therefore, for any object V of $\mathcal{C}(\mathfrak{p}, \mathfrak{l})$, we have the vector space isomorphism

$$\mathrm{Hom}_{\mathrm{U}_\mathrm{q}(\mathfrak{p})}\left(\mathrm{U}_\mathrm{q}(\mathfrak{g}_{\leq 0}), \mathrm{V}\right) \cong \mathrm{Hom}_{\mathrm{U}_\mathrm{q}(\mathfrak{p}_0)}\left(\mathrm{U}_\mathrm{q}(\mathfrak{g}_0), \mathcal{F}^{\mathfrak{p}_0,\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}(\mathrm{V})\right). \tag{5.3}$$

Denote by $P : \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p})}\left(\mathrm{U}_q(\mathfrak{g}_{\leq 0}), V\right) \rightarrow \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{p}_0)}\left(\mathrm{U}_q(\mathfrak{g}_0), V\right)$ the map induced by the inclusion of $\mathrm{U}_q(\mathfrak{g}_0)$ in $\mathrm{U}_q(\mathfrak{g}_{\leq 0})$,

$$\langle P(\zeta), x\rangle = \langle \zeta, x\rangle, \quad \forall x \in \mathrm{U}_q(\mathfrak{g}_0) \subset \mathrm{U}_q(\mathfrak{g}_{\leq 0}).$$

This map is $\mathrm{U}_q(\mathfrak{g}_0)$-equivariant, as for any $u \in \mathrm{U}_q(\mathfrak{g}_0)$, we have

$$\langle P(u \circ \zeta), x\rangle = (-1)^{[u]([x]+[\zeta])}\langle \zeta, xu\rangle = \langle u \circ P(\zeta), x\rangle.$$

Now every element in $\mathrm{U}_q(\mathfrak{g}_{\leq 0})$ may be expressed in the form $\sum y_i u_i$ with $y_i \in \mathrm{U}_q(\mathfrak{p})$ and $u_i \in \mathrm{U}_q(\mathfrak{g}_0)$. We have $\langle \zeta, \sum y_i u_i\rangle = \sum (-1)^{[y_i][\zeta]}\pi_V(y_i)\langle P(\zeta), u_i\rangle$. Thus $P(\zeta) = 0$ if and only if $\zeta = 0$. Therefore, the $\mathrm{U}_q(\mathfrak{g}_0)$-map P is injective, which must be bijective because of the vector space isomorphism (5.3).

Since $\mathrm{U}_q(\mathfrak{l}) \subset \mathrm{U}_q(\mathfrak{p}_0) \subset \mathrm{U}_q(\mathfrak{g}_0)$, and $\mathrm{U}_q(\mathfrak{l}) \subset \mathrm{U}_q(\mathfrak{p}) \subset \mathrm{U}_q(\mathfrak{g}_{\leq 0})$, by (4.3) we have

$$\begin{aligned}
\mathrm{I}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{p}_0,\mathfrak{l}} \circ \mathcal{F}^{\mathfrak{p}_0,\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}(V) &= \mathrm{Hom}_{\mathrm{U}_\mathrm{q}(\mathfrak{p}_0)}\left(\mathrm{U}_\mathrm{q}(\mathfrak{g}_0), \mathcal{F}^{\mathfrak{p}_0,\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}(\mathrm{V})\right)[\mathrm{U}_q(\mathfrak{l})],\\
\mathrm{I}^{\mathfrak{g}_{\leq 0},\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}(V) &= \mathrm{Hom}_{\mathrm{U}_\mathrm{q}(\mathfrak{p})}\left(\mathrm{U}_\mathrm{q}(\mathfrak{g}_{\leq 0}), \mathrm{V}\right)[\mathrm{U}_q(\mathfrak{l})].
\end{aligned}$$

The restriction of the $\mathrm{U}_q(\mathfrak{g}_0)$-equivariant map P to $\mathrm{I}^{\mathfrak{g}_{\leq 0},\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}(V)$ now leads to the sought after $\mathrm{U}_q(\mathfrak{g}_0)$-module isomorphism

$$\mathcal{F}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{g}_{\leq 0},\mathfrak{l}} \circ \mathrm{I}^{\mathfrak{g}_{\leq 0},\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}(V) \cong \mathrm{I}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{p}_0,\mathfrak{l}} \circ \mathcal{F}^{\mathfrak{p}_0,\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}(V). \tag{5.4}$$

□

We also have the following easy result.

Lemma 5.2. *The functor* $\mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\mathfrak{g}_{\leq 0},\mathfrak{g}_0} : \mathcal{C}(\mathfrak{g}_{\leq 0}, \mathfrak{g}_0) \to \mathcal{C}(\mathfrak{g}, \mathfrak{g})$ *is exact with*

$$\mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\mathfrak{g}_{\leq 0},\mathfrak{g}_0}(V) \quad = \quad \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g}_{\leq 0})}\left(\mathrm{U}_q(\mathfrak{g}),\, V\right),$$

for any object V *in* $\mathcal{C}(\mathfrak{g}_{\leq 0}, \mathfrak{g}_0)$.

Proof. We need to show that $\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g}_{\leq 0})}\left(\mathrm{U}_q(\mathfrak{g}),\, \cdot\,\right)$ is exact on $\mathcal{C}(\mathfrak{g}_{\leq 0}, \mathfrak{g}_0)$, and for any V in $\mathcal{C}(\mathfrak{g}_{\leq 0}, \mathfrak{g}_0)$, $\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g}_{\leq 0})}\left(\mathrm{U}_q(\mathfrak{g}),\, V\right)$ is $\mathrm{U}_q(\mathfrak{g}_0)$-finite. It is fairly easy to see that $\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g}_{\leq 0})}\left(\mathrm{U}_q(\mathfrak{g}),\, V\right)$ is spanned by integral weigh vectors since V is an object of $\mathcal{C}(\mathfrak{g}_{\leq 0}, \mathfrak{g}_0)$. Let $\mathcal{U}^{+1}$ denote the subspace of $\mathrm{U}_q(\mathfrak{g})$ spanned by the ordered products of $(E_{i\alpha})^{\theta_{i\alpha}}$, $i \leq m < \alpha$, $\theta_{i\alpha} = 0, 1$. Clearly $\dim \mathcal{U}^{+1} = 2^{mn}$. Then

$$\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g}_{\leq 0})}\left(\mathrm{U}_q(\mathfrak{g}),\, V\right) \cong \left(\mathcal{U}^{+1}\right)^* \otimes V.$$

This in particular implies that $\mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g}_{\leq 0})}\left(\mathrm{U}_q(\mathfrak{g}),\, \cdot\,\right)$ is exact. Given any $x \in \mathrm{U}_q(\mathfrak{g}_0)$ and $\eta \in \mathcal{U}^{+1}$, there exist $\eta_j \in \mathcal{U}^{+1}$ and $x_j \in \mathrm{U}_q(\mathfrak{g}_0)$ such that $\eta x = \sum x_j \eta_j$. Let $\zeta = \sum f_i \otimes v_i$ be in $(\mathcal{U}^{+1})^* \otimes V$. We have

$$\begin{aligned}\langle x \circ \zeta, \eta\rangle &= \sum (-1)^{[v_i][\eta]} \langle f_i, \eta x\rangle v_i \\ &= \sum (-1)^{[v_i][\eta]} \langle f_i, \eta_j\rangle \pi_V(x_j) v_i.\end{aligned}$$

Since V is $\mathrm{U}_q(\mathfrak{g}_0)$-finite, we can deduce from this equation that $\mathrm{U}_q(\mathfrak{g}_0) \circ \zeta$ is finite dimensional for any $\zeta = \sum f_i \otimes v_i \in (\mathcal{U}^{+1})^* \otimes V$. This completes the proof. □

The following proposition is one of the main results of this paper.

Proposition 5.1. *Let* $\mathrm{U}_q(\mathfrak{l}) \subseteq \mathrm{U}_q(\mathfrak{g}_0)$ *be a reductive quantum subalgebra of* $\mathrm{U}_q(\mathfrak{g})$. *Let* $\mathrm{U}_q(\mathfrak{p}) \supseteq \mathrm{U}_q(\bar{\mathfrak{b}})$ *be the parabolic quantum sub-superalgebra of* $\mathrm{U}_q(\mathfrak{g})$ *with* $\mathrm{U}_q(\mathfrak{l})$ *as its Levi factor. Let* $L_\lambda^{(\mathfrak{p})}$ *be a finite dimensional irreducible* $\mathrm{U}_q(\mathfrak{p})$*-module with* $\mathrm{U}_q(\mathfrak{l})$*-highest weight* $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$. *Denote by* $L_\lambda^{(\mathfrak{p}_0)}$ *the natural restriction of* $L_\lambda^{(\mathfrak{p})}$ *to a* $\mathrm{U}_q(\mathfrak{p}_0)$*-module. Then*

$$H^{0,k}(G/P, \mathcal{S}(L_\lambda^{(\mathfrak{p})})) \quad = \quad \mathrm{Hom}_{\mathrm{U}_q(\mathfrak{g}_{\leq 0})}(\mathrm{U}_q(\mathfrak{g}),\, \left(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}}\right)^k (L_\lambda^{(\mathfrak{p}_0)})), \tag{5.5}$$

where $\left(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}}\right)^k (L_\lambda^{(\mathfrak{p}_0)})$ *is regarded as a* $\mathrm{U}_q(\mathfrak{g}_{\leq 0})$*-module with* $E_{m+1,m}$ *acting by zero.*

Proof. By Lemma 4.3, $\mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\mathfrak{p},\mathfrak{l}} = \mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\mathfrak{g}_{\leq 0},\mathfrak{g}_0} \circ \mathrm{I}^{\mathfrak{g}_{\leq 0},\mathfrak{g}_0}_{\mathfrak{g}_{\leq 0},\mathfrak{l}} \circ \mathrm{I}^{\mathfrak{g}_{\leq 0},\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}$. By Lemma 5.1,

$$\mathcal{F}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_{\leq 0},\mathfrak{g}_0} \circ \mathrm{I}^{\mathfrak{g}_{\leq 0},\mathfrak{g}_0}_{\mathfrak{g}_{\leq 0},\mathfrak{l}} \circ \mathrm{I}^{\mathfrak{g}_{\leq 0},\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}} = \mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_0,\mathfrak{l}} \circ \mathrm{I}^{\mathfrak{g}_0,\mathfrak{l}}_{\mathfrak{p}_0,\mathfrak{l}} \circ \mathcal{F}^{\mathfrak{p}_0,\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}.$$

Using Lemma 4.3 again, we obtain

$$\mathcal{F}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_{\leq 0},\mathfrak{g}_0} \circ \mathrm{I}^{\mathfrak{g}_{\leq 0},\mathfrak{g}_0}_{\mathfrak{p},\mathfrak{l}} \quad = \quad \mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}} \circ \mathcal{F}^{\mathfrak{p}_0,\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}. \tag{5.6}$$

Recall the following elementary facts: Let $\mathcal{C} \xrightarrow{G} \mathcal{C}'$ be a left exact covariant functor. (a). Suppose $\mathcal{C}' \xrightarrow{F} \mathcal{C}''$ is an exact covariant functor. Then $F \circ G$ is left exact, and its right derive functors are $(F \circ G)^k = F \circ (G)^k$. (b). Suppose $\tilde{\mathcal{C}} \xrightarrow{F} \mathcal{C}'$ is an exact covariant functor. Then $G \circ F$ is left exact, and its right derived functors are $(G \circ F)^k = (G)^k \circ F$. Applying these results to the situation at hand, we arrive at

$$\mathcal{F}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_{\leq 0},\mathfrak{g}_0} \circ \left(\mathrm{I}^{\mathfrak{g}_{\leq 0},\mathfrak{g}_0}_{\mathfrak{p},\mathfrak{l}}\right)^k (L_\lambda^{(\mathfrak{p})}) \quad = \quad \left(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}}\right)^k (L_\lambda^{(\mathfrak{p}_0)}).$$

The derived functor $\left(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}}\right)^k$ on the right hand side can be computed by using the quantum Bott-Borel-Weil theorem [1] for $\mathrm{U}_q(\mathfrak{g}_0) = \mathrm{U}_q(\mathfrak{gl}_m) \otimes \mathrm{U}_q(\mathfrak{gl}_n)$. Now $\left(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}}\right)^k (L_\lambda^{(\mathfrak{p}_0)})$

is either zero or a finite dimensional irreducible $\mathrm{U}_q(\mathfrak{g}_0)$-module. Therefore, its inverse image under the forgetful functor $\mathcal{F}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{g}_{\leq 0},\mathfrak{g}_0}$ must be either zero or $\mathrm{U}_q(\mathfrak{g}_{\leq 0})$-irreducible. In both cases, $E_{m+1,m}$ acts by zero. Thus by using Lemma 5.2, we have

$$\left(\mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\mathfrak{p},\mathfrak{l}}\right)^k (L^{(\mathfrak{p})}_\lambda) \ = \ \mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\mathfrak{g}_{\leq 0},\mathfrak{g}_0}\left(\left(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}}\right)^k (L^{(\mathfrak{p}_0)}_\lambda)\right),$$

where $\left(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}}\right)^k (L^{(\mathfrak{p}_0)}_\lambda)$ is regarded as a $\mathrm{U}_q(\mathfrak{g}_{\leq 0})$-module with $E_{m+1,m}$ acting by zero. Another easy application of Lemma 5.2 completes the proof. □

By using the proposition we can easily prove the following result.

Theorem 5.2. *Let $\mathrm{U}_q(\mathfrak{l}) \subseteq \mathrm{U}_q(\mathfrak{g}_0)$ be a reductive quantum subalgebra of $\mathrm{U}_q(\mathfrak{g})$. Let $\mathrm{U}_q(\mathfrak{p}) \supseteq \mathrm{U}_q(\bar{\mathfrak{b}})$ be the parabolic quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$ with $\mathrm{U}_q(\mathfrak{l})$ as its Levi factor. Let $L^{(\mathfrak{p})}_\lambda$ be a finite dimensional irreducible $\mathrm{U}_q(\mathfrak{p})$-module with $\mathrm{U}_q(\mathfrak{l})$-highest weight $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$.*

(1) *If λ is $\mathfrak{g}$-regular, then there exists a unique element w of the Weyl group of $\mathfrak{g}_0$ rendering $\mu := w(\lambda+\rho)-\rho$ dominant with respect to $\mathfrak{g}$. In this case,*

$$H^{0,k}(G/P, \mathcal{S}(L^{(\mathfrak{p})}_\lambda)) \ = \ \begin{cases} K^{(\mathfrak{g})}_\mu, & k = |w|, \\ 0, & k \neq |w|, \end{cases}$$

where $|w|$ denotes the length of w.

(2) *If λ is not $\mathfrak{g}$-regular, then*

$$H^{0,k}(G/P, \mathcal{S}(L^{(\mathfrak{p})}_\lambda)) \ = \ 0, \quad \forall k.$$

Proof. According to the quantum Bott-Borel-Weil theorem for quantized universal enveloping algebras of ordinary Lie algebras [1], the $\left(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}}\right)^k (L^{(\mathfrak{p}_0)}_\lambda)$ vanishes for all k if λ is not $\mathfrak{g}_0$-regular. If λ is $\mathfrak{g}_0$-regular, then $\left(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}}\right)^k (L^{(\mathfrak{p}_0)}_\lambda)$ is concentrated at on degree, namely, $\left(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}}\right)^k (L^{(\mathfrak{p}_0)}_\lambda)$ is non-vanishing for one k only. We have

$$\left(\mathrm{I}^{\mathfrak{g}_0,\mathfrak{g}_0}_{\mathfrak{p}_0,\mathfrak{l}}\right)^{|w|} (L^{(\mathfrak{p}_0)}_\lambda) \ = \ L^{(\mathfrak{g}_0)}_\mu,$$

where $L^{(\mathfrak{g}_0)}_\lambda$ is the irreducible $\mathrm{U}_q(\mathfrak{g}_0)$-module with highest weight μ. Using this result in Proposition 5.1, we arrive at the Theorem. □

Remark 5.2. *An easy examination will show that the proof for Proposition 5.1 still goes through for $\mathrm{U}_q\left(\mathfrak{gl}_{m_1|n_1} \oplus \mathfrak{gl}_{m_2|n_2} \oplus ... \oplus \mathfrak{gl}_{m_i|n_i}\right)$ for any finite i. The same comment applies to Theorem 5.2.*

5.2.2. *The general case.* We investigate the general case in this subsection. Now $\mathrm{U}_q(\mathfrak{l})$ is an arbitrary reductive quantum sub-superalgebra of $\mathrm{U}_q(\mathfrak{g})$, and $\mathrm{U}_q(\mathfrak{p})$ is the parabolic containing $\mathrm{U}_q(\bar{\mathfrak{b}})$ and has the Levi factor $\mathrm{U}_q(\mathfrak{l})$. Let $\mathrm{U}_q(\bar{\mathfrak{b}}_\mathfrak{l}) = \mathrm{U}_q(\bar{\mathfrak{b}}) \cap \mathrm{U}_q(\mathfrak{l})$ be the Borel subalgebra of $\mathrm{U}_q(\mathfrak{l})$. Denote by

$$\mathcal{F}^{\mathfrak{l},\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}} : \mathcal{C}(\mathfrak{p},\mathfrak{l}) \to \mathcal{C}(\mathfrak{l},\mathfrak{l}), \qquad \mathcal{F}^{\bar{\mathfrak{b}}_\mathfrak{l},\mathfrak{h}}_{\bar{\mathfrak{b}},\mathfrak{h}} : \mathcal{C}(\bar{\mathfrak{b}},\mathfrak{h}) \to \mathcal{C}(\bar{\mathfrak{b}}_\mathfrak{l},\mathfrak{h})$$

the forgetful functors. We have the following result.

Lemma 5.3. $\mathcal{F}^{\mathfrak{l},\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}} \circ \mathrm{I}^{\mathfrak{p},\mathfrak{l}}_{\bar{\mathfrak{b}},\mathfrak{h}} = \mathrm{I}^{\mathfrak{l},\mathfrak{l}}_{\bar{\mathfrak{b}}_\mathfrak{l},\mathfrak{h}} \circ \mathcal{F}^{\bar{\mathfrak{b}}_\mathfrak{l},\mathfrak{h}}_{\bar{\mathfrak{b}},\mathfrak{h}}$.

Proof. The proof is much the same as that for (5.2). Because of the given conditions on $\mathrm{U}_q(\mathfrak{p})$ and $\mathrm{U}_q(\mathfrak{l})$, equation (4.3) gives

$$\mathrm{I}^{\mathfrak{p},\mathfrak{l}}_{\bar{\mathfrak{b}},\mathfrak{h}}(V) \ = \ \mathrm{Hom}_{\mathrm{U}_q(\bar{\mathfrak{b}})}\left(\mathrm{U}_q(\mathfrak{p}),\, V\right)[\mathrm{U}_q(\mathfrak{l})],$$

for any object V of $\mathcal{C}(\bar{\mathfrak{b}}, \mathfrak{h})$. We can easily show that there exists the even $\mathrm{U}_q(\mathfrak{l})$-module isomorphism

$$P: \mathrm{Hom}_{\mathrm{U}_q(\bar{\mathfrak{b}})}(\mathrm{U}_q(\mathfrak{p}), V) \stackrel{\sim}{\longrightarrow} \mathrm{Hom}_{\mathrm{U}_q(\bar{\mathfrak{b}}_{\mathfrak{l}})}(\mathrm{U}_q(\mathfrak{l}), V)$$

defined by $\langle \zeta, ux\rangle = \pi_V(u)\langle P(\zeta), x\rangle$, for all $u \in \mathrm{U}_q(\bar{\mathfrak{b}})$, $x \in \mathrm{U}_q(\mathfrak{l})$. Therefore,

$$\mathrm{I}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\mathfrak{p},\mathfrak{l}}(V) = \mathrm{Hom}_{\mathrm{U}_q(\bar{\mathfrak{b}}_{\mathfrak{l}})}(\mathrm{U}_q(\mathfrak{l}), V)[\mathrm{U}_q(\mathfrak{l})].$$

On the other hand,

$$\mathrm{I}_{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}}^{\mathfrak{l},\mathfrak{l}} \circ \mathcal{F}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}}(V) = \mathrm{Hom}_{\mathrm{U}_q(\bar{\mathfrak{b}}_{\mathfrak{l}})}\left(\mathrm{U}_q(\mathfrak{l}), \mathcal{F}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}}(V)\right)[\mathrm{U}_q(\mathfrak{l})].$$

Thus the claim of the Lemma is indeed true for any object of $\mathcal{C}(\bar{\mathfrak{b}}, \mathfrak{h})$. The claim also clearly holds true for morphisms of $\mathcal{C}(\bar{\mathfrak{b}}, \mathfrak{h})$. □

Theorem 5.3. *Let $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$ be $\mathfrak{l}$-dominant. Inflate $K_\lambda^{(\mathfrak{l})}$ to a $\mathrm{U}_q(\mathfrak{p})$ module by requiring that all the generators of $\mathrm{U}_q(\mathfrak{p})$ not contained in $\mathrm{U}_q(\mathfrak{l})$ act by zero, and denote the resultant $\mathrm{U}_q(\mathfrak{p})$-module by $K_\lambda^{(\mathfrak{p})}$.*

(1) *If λ is $\mathfrak{g}$-regular, then there exists a unique w in the Weyl group of $\mathfrak{g}_0$ rendering $\mathfrak{g}$-dominant the following weight $\mu := w(\lambda+\rho)-\rho$. In this case,*

$$H^{0,k}(G/P, \mathcal{S}(K_\lambda^{(\mathfrak{p})})) = \begin{cases} K_\mu^{(\mathfrak{g})}, & k = |w|, \\ 0, & k \neq |w|. \end{cases}$$

(2) *If λ is not $\mathfrak{g}$-regular, then $H^{0,k}(G/P, \mathcal{S}(K_\lambda^{(\mathfrak{p})})) = 0, \quad \forall k.$*

Proof. We use Lemma 4.3 to write $\mathrm{I}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\mathfrak{g},\mathfrak{g}} = \mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}} \circ \mathrm{I}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\mathfrak{p},\mathfrak{l}}$. The functor $\mathrm{I}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\mathfrak{p},\mathfrak{l}}$ takes injectives to injectives. Thus for an irreducible $\mathrm{U}_q(\bar{\mathfrak{b}})$-module $\mathbb{C}(q)_\lambda$ with an arbitrary weight $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, we have a first quadrant spectral sequence, the Grothendieck spectral sequence (Sections 5.8 and 10.8 of [19]),

$$E_r^{p,q} \Longrightarrow \left(\mathrm{I}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\mathfrak{g},\mathfrak{g}}\right)^{p+q}(\mathbb{C}(q)_\lambda),$$

with $E_2^{p,q}$ term

$$E_2^{p,q} = \left(\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}\right)^p \left(\mathrm{I}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\mathfrak{p},\mathfrak{l}}\right)^q(\mathbb{C}(q)_\lambda),$$

where the differential on $E_r^{p,q}$ has bi-degree $(r,\ 1-r)$. We shall prove below that $\left(\mathrm{I}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\mathfrak{p},\mathfrak{l}}\right)^q(\mathbb{C}(q)_\lambda)$ is concentrated at one degree. Let us take this as granted for the moment. Then the spectral sequence collapses at E_2, and we obtain

$$\left(\mathrm{I}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\mathfrak{g},\mathfrak{g}}\right)^{p+q}(\mathbb{C}(q)_\lambda) = \left(\mathrm{I}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{g},\mathfrak{g}}\right)^p \left(\mathrm{I}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\mathfrak{p},\mathfrak{l}}\right)^q(\mathbb{C}(q)_\lambda). \tag{5.7}$$

Now we consider $\left(\mathrm{I}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\mathfrak{p},\mathfrak{l}}\right)^q(\mathbb{C}(q)_\lambda)$ for arbitrary $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$. By Lemma 5.3, we have

$$\mathcal{F}_{\mathfrak{p},\mathfrak{l}}^{\mathfrak{l},\mathfrak{l}} \circ \left(\mathrm{I}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\mathfrak{p},\mathfrak{l}}\right)^q(\mathbb{C}(q)_\lambda) = \left(\mathrm{I}_{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}}^{\mathfrak{l},\mathfrak{l}}\right)^q \circ \mathcal{F}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}}(\mathbb{C}(q)_\lambda). \tag{5.8}$$

Note that $\mathrm{U}_q(\mathfrak{l})$ is the tensor product of the quantized universal enveloping algebra of the direct sum of some general linear algebras and possibly also a general linear superalgebra. By Theorem 5.2 and Remark 5.2, the right hand side is zero unless λ is $\mathfrak{l}$-regular. When λ is $\mathfrak{l}$-regular, $\left(\mathrm{I}_{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}}^{\mathfrak{l},\mathfrak{l}}\right)^q \circ \mathcal{F}_{\bar{\mathfrak{b}},\mathfrak{h}}^{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}}(\mathbb{C}(q)_\lambda)$ is concentrated at one degree.

Explicitly, there exists a unique $w_{\mathfrak{l}}$ in the Weyl group of $\mathfrak{l}$ rendering $w_{\mathfrak{l}}(\lambda+\rho_{\mathfrak{l}})-\rho_{\mathfrak{l}}$ dominant with respect to $\mathfrak{l}$, and we have

$$\left(\mathrm{I}^{\mathfrak{l},\mathfrak{l}}_{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}}\right)^{|w_{\mathfrak{l}}|}\circ\mathcal{F}^{\bar{\mathfrak{b}}_{\mathfrak{l}},\mathfrak{h}}_{\mathfrak{b},\mathfrak{h}}(\mathbb{C}(q)_\lambda)=K^{(\mathfrak{l})}_{w_{\mathfrak{l}}(\lambda+\rho_{\mathfrak{l}})-\rho_{\mathfrak{l}}}.$$

Here $\rho_{\mathfrak{l}}$ is half of the signed-sum of the positive roots of $\mathfrak{l}$ relative to $\mathfrak{b}_{\mathfrak{l}}=\mathfrak{b}\cap\mathfrak{l}$. Needless to say, the formula remains valid if we replace $\rho_{\mathfrak{l}}$ by ρ.

In order to determine $\left(\mathrm{I}^{\mathfrak{p},\mathfrak{l}}_{\bar{\mathfrak{b}},\mathfrak{h}}\right)^{q}(\mathbb{C}(q)_\lambda)$, we consider all the possible objects W_λ of $\mathcal{C}(\mathfrak{p},\mathfrak{l})$ satisfying $\mathcal{F}^{\mathfrak{l},\mathfrak{l}}_{\mathfrak{p},\mathfrak{l}}(W_\lambda)=K^{(\mathfrak{l})}_{w_{\mathfrak{l}}(\lambda+\rho)-\rho}$. Any two weights of $K^{(\mathfrak{l})}_{w_{\mathfrak{l}}(\lambda+\rho)-\rho}$ can only differ by an integeral combination of the roots of $\mathfrak{l}$. This in particular requires that all the generators of $\mathrm{U}_q(\mathfrak{p})$ not contained in $\mathrm{U}_q(\mathfrak{l})$ act on W_λ by zero. Therefore,

$$\left(\mathrm{I}^{\mathfrak{p},\mathfrak{l}}_{\bar{\mathfrak{b}},\mathfrak{h}}\right)^{|w_{\mathfrak{l}}|}(\mathbb{C}(q)_\lambda)\quad=\quad K^{(\mathfrak{p})}_{w_{\mathfrak{l}}(\lambda+\rho)-\rho}.$$

By using the given condition that λ is $\mathfrak{l}$-dominant, we obtain from (5.7)

$$\left(\mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\mathfrak{p},\mathfrak{l}}\right)^{k}\left(K^{(\mathfrak{p})}_{\lambda}\right)\quad=\quad\left(\mathrm{I}^{\mathfrak{g},\mathfrak{g}}_{\bar{\mathfrak{b}},\mathfrak{h}}\right)^{k}(\mathbb{C}(q)_\lambda).$$

Using the special case of Theorem 5.2 with the parabolic being $\mathrm{U}_q(\bar{\mathfrak{b}})$, we complete the proof. □

References

[1] Andersen, H.H.; Polo, P; Wen, K. X. Representations of quantum algebras. Invent. Math. 104 (1991), no. 1, 1–59.

[2] Bott, R. Homogeneous vector bundles. Ann. of Math. (2) 66 (1957), 203–248.

[3] A. Connes, *Noncommutative geometry*, Academic Press (1994).

[4] Enright, T. J.; Wallach, N. R. Notes on homological algebra and representations of Lie algebras. Duke Math. J. 47 (1980), no. 1, 1–15.

[5] Gover, A. R.; Zhang, R. B. Geometry of quantum homogeneous vector bundles and representation theory of quantum groups. I. Rev. Math. Phys. 11 (1999), no. 5, 533–552.

[6] Jimbo, M. A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation. Lett. Math. Phys. 10 (1985), no. 1, 63–69.

[7] V. G. Kac, *Lie superalgebras*, Adv. Math. **26** (1977) 8-96.

[8] A.W. Knapp and D.A. Vogan, *Cohomological induction and unitary representations*. Princeton Mathematical Series, **45**. Princeton University Press, Princeton, NJ, (1995).

[9] Khoroshkin, S. M.; Tolstoy, V. N. *Universal R-matrix for quantized (super)algebras*. Comm. Math. Phys. 141 (1991), no. 3, 599–617.

[10] Yu. I. Manin, *Gauge field theory and complex geometry*, 2nd edition, Springer-Verlag, Berlin (1997).

[11] Yu. I. Manin, *Quantum groups and noncommutative geometry*. Universite de Montreal, Centre de Recherches Mathematiques, Montreal, QC, 1988.

[12] Yu. I. Manin, *Multiparametric quantum deformation of the general linear supergroup*. Comm. Math. Phys. **123** (1989) 163–175.

[13] Milnor, J. W.; Moore, J. C. *On the structure of Hopf algebras*, Ann. of Math. **81** (1965) 211-264.

[14] Montgomery, S. Hopf algebras and their actions on rings, Regional Conference Series in Math. No. **82**, Amer. Math. Soc., Providence, R.I. (1993).

[15] Penkov, I. B. Borel-Weil-Bott theory for classical Lie superalgebras, J. Sov. Math **51** (1990) 2108-2140.

[16] Santos, J. Foncteurs de Zuckerman pour les superalgebres de Lie. J. Lie Theory 9 (1999), no. 1, 69–112.

[17] Scheunert, M. The theory of Lie superalgebras. Lecture Notes Math. **716**, Springer-Verlag, Berlin (1979).

[18] M. Scheunert and R. B. Zhang, *The general linear supergroup and its Hopf superalgebra of regular functions*, J. Algebra **254** (2002) 44-83.

[19] C.A. Weibel, *An introduction to homological algebra*, Cambridge University Press (1994).
[20] Zhang, R. B. Finite-dimensional irreducible representations of the quantum supergroup $\mathrm{U}_q(\mathrm{gl}(m/n))$. J. Math. Phys. 34 (1993), no. 3, 1236–1254.
[21] Zhang, R. B. Structure and representations of the quantum general linear supergroup. Commun. Math. Phys. **195** (1998) 525-547.
[22] Zhang, R. B.; Zou, Y. M. Spherical Functions On Symmetric Superspaces. Preprint.
[23] Zou, Y. M. Deformations of enveloping algebra of Lie superalgebra $\mathrm{sl}(m,n)$. Comm. Math. Phys. 162 (1994), no. 3, 467–479.

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